협의의 의사함수에 관한 연구

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A Study on Strictly pseudoconvex functions

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요　약

본 논문에서는 함수 f의 구배에 직교하는 부분 공간에서, 함수 f의 헤시안 행열이 양의 반정부호이면 함수 f는 준요함수임을 밝히고, 나아가서 준요함수의 협의의 의사성에 대한 일차판정법을 얻는다.

○ Introduction

Quasiconvexity is the most promising and useful generalization of convexity. It occurs in many problems of mathematical programming and economics where convexity is too restrictive condition. And strict pseudoconvexity is very important generalization of strict convexity.

In this paper, we investigate several properties on quasiconvex, strictly pseudoconuex functions. In section 1, we obstain the fact that if the Hessian matrix $H(x)$ of f is positive semidefinite in the subspace orthogonal to $\nabla f(x)$, then f is quasi-convex. In section 2, we obtain a first order criterion for strict pseudoconvexity of quasiconvex functions.

1. Quasiconvex Functions

We begin by the following definition.

Definition 1.1 Let $f: s \rightarrow E$, where S is a nonempty convex set in En. The function f is said to be quasiconvex if for each x_1 and $x_2 \in s$, the following inequality is true: $f[\lambda x_1 + (1-\lambda)x_2] \leq \max\{f(x_1), f(x_2)\}$, for each $\lambda \in (0, 1)$

Remark : Let S be a nonempty open convex set in En, and let $f: s \rightarrow E_1$ be differentiable on S. Then f is quasiconvex if and only if either one of the following equivalent statements hold.

1. It $x_1, x_2 \in s$ and $f(x_1) \leq f(x_2)$, then $f(x_2)^t (x_1 - x_2) \leq 0$
2. If $x_1, x_2 \in s$ and $\nabla f(x_2)^t (x_1 - x_2) > 0$, then $f(x_1) > f(x_2)$

Theorem 1.1 Let S be a nonempty open convex set in En and $f: s \rightarrow E_1$. Then f is quasiconvex on s if and only if $f_{xo,h}$ is quasiconvex for all $xo, h \in En$, where $f_{xo,h}(t) = f(xo + th)$, $t \in E_1$, $x_0 + th \in S$.
i.e., the restriction of f to any line segment in S is quasiconvex.

(proof) (⇒) Let $xo, h \in En$ and $\lambda \in (0, 1)$

Let $t_1, t_2 \in E_1$ such that $xo + t_1 h \in s$ and $xo + t_2 h \in s$

$$\begin{aligned} f_{xo,h}[\lambda t_1 + (1-\lambda)t_2] &= f[xo + \{\lambda t_1 + (1-\lambda)t_2\}h] \\ &= f[xo + \lambda t_1 h + (1-\lambda)t_2 h] \\ &= f[\lambda(xo + t_1 h) + (1-\lambda)(xo + t_2 h)] \\ &\leq \max\{f(xo + t_1 h), f(xo + t_2 h)\} \\ &= \max\{f_{xo,h}(t_1), f_{xo,h}(t_2)\} \end{aligned}$$

Hence, $f_{xo,h}[\lambda t_1 + (1-\lambda)t_2] \leq \max\{f_{xo,h}(t_1), f_{xo,h}(t_2)\}$

Therefore, $f_{xo,h}$ is quasiconvex.

(⇐) Let $x, y \in s$ and $\lambda \in (0, 1)$

By assumption, $f_{y, x-y}$ is quasiconvex.

$$f[\lambda x+(1-\lambda)y] = f_{y,\,x-y}(\lambda)$$
$$= f_{y,\,x-y}[\lambda .1+(1-\lambda).0]$$
$$\leq \max\{f_{y,\,x-y}(1),\ f_{y,\,x-y}(0)\}$$
$$= \max\{f[y+1.(x-y)],\ f[y+0.(x-y)]\}$$
$$= \max\{f(x),\ f(y)\}$$

Hence, $f[\lambda x+(1-\lambda)y] \leq \max\{f(x), f(y)\}$

Therefore, f is quasiconvex.

Theorem 1. 2 Let S be an open convex set in En and $f: s \rightarrow E_1$ be a twice differentiable function.

Then, f is quasiconvex on S if for all $x \epsilon s$, $\nabla f(x)^t h = 0$ implies $h^t H(x) h \geq 0$, where $H(x)$ is the Hessian matrix of f at $x \epsilon s$.

(proof) Suppose that there exists an $(x, h) \epsilon sxEn$ such that $\nabla f(x)^t h = 0$ and $h^t H(x) h < 0$.

Let $(t) = f(x+th)$ for any $t \epsilon \{t \epsilon E : x+th \epsilon s\}$

By theorem 1. 1, is quasiconvex.

Since f is twice differentiable, θ is twice differentiable.

$f(x)^t h = \theta'(0)$ and $h^t \ \ f(x)^t h = \theta''(0)$

Since θ is twice differentiable, $\theta(t) = \theta(0) + t\theta'(0) + \frac{1}{2}t^2\theta''(0) + t^2\varepsilon(t)$

$$= (0) + \frac{1}{2}t^2\theta''(0) + t^2\varepsilon(t)$$

, where $\lim_{t \rightarrow 0} \varepsilon(t) = 0$

Since $\theta''(0) < 0$ and $\lim_{t \rightarrow 0} \varepsilon(t) = 0$, there exists $\delta > 0$ such that for $0 < |t| < \delta$,

$\frac{1}{2}\theta''(0) < \varepsilon(t) < -\frac{1}{2}\theta''(0)$.

For $0 < |t| < \delta$, $\theta(t) - \theta(0) < \frac{1}{2}t^2\theta''(0) + t^2(-\frac{1}{2}\theta''(0)) = 0$

Hence, $\theta(t) < \theta(0)$ (1. 1)

Let $0 < |t| < \delta$.

By the quasiconvexity of f, $\theta(0) = \theta[\frac{1}{2}t_1 + (1-\frac{1}{2})t_1]$

$$\leq \max\{\theta(t_1), \theta(-t)\}$$

This contradicts to (1. 1)

Hence, $f(t)^t h = 0$ implies $h^t H(x) \ h \geq 0$.

2. Strictly pseudoconvex functions.

Definition 2. 1. Let s be a nonempty open set in En and let $f: s \to E_1$ be differentiable.

The function f is said to be strictly pseudoconvex on s if for each distinct $x_1, x_2 \in s$, $\nabla f(x)^t (x_1 - x_2) \geq 0$ implies that $f(x_2) > f(x_1)$

Theorem 2. 1. Let s be on open convex set in En and $f: s \to E_1$ be differentiable. Then, f is strictly pseudoconvex on s if and only if $f_{xo,h}$ is strictly pseudoconvex for all $xo, h \in En$, where $f_{xo,h}(t) = f(xo + th)$, $t \in E_1$, $xo + th \in s$.

(proof) ($\Rightarrow$) Let xo, $h \in En$

Let $t_1, t_2 \in E_1$ $(t_1 \neq t_2)$ such that $xo + t_1 h \in s$ and $xo + t_2 h \in s$

Assume that $f_{xo,h}(t_2) \leq f_{xo,h}(t_1)$

By the definition of $f_{xo,h}$, $f(xo + t_2 h) \leq f(xo + t_1 h)$

By the strict pseudoconvexity of f, $f(xo + t_1 h)^t (t_2 - t_1) h < 0$.

Since $\nabla f(xo + t_1 h)^t h = f'_{xo,h}(t_1)$, $f'_{xo,h}(t_1)(t_2 - t_1) < 0$

Hence, $f_{xo,h}$ is strictly pseudoconvex.

($\Leftarrow$) Let $x, y \in s$ $(x \neq y)$

Suppose that $f(y) \leq f(x)$

By assumption, $f_{y,x-y}$ is strictly pseudoconvex.

$f(y) = f_{y,x-y}(0) \leq f(x) = f_{y,x-y}(1)$

By the strict pseudoconvexity of $f_{y,x-y}$, $f'_{y,x-y}(1) > 0$

Since $\nabla f[y + (x-y)]^t (x-y) = f'_{y,x-y}(1)$, $\nabla f(x)^t (y-x) < 0$

Hence f is strictly pseudoconvex.

Theorem 2. 2. Let s be an open convex set in En and $f: s \to E_1$ be differentiable. Let f be strictly pseudoconvex on s. If $\nabla f(x)^t (y-x) \geq 0$, for all $y \in s$, then x is the unique global minimum of f over s.

(proof) Since $\nabla f(x)^t (y-x) \geq 0$ for all $y \in s$, by the strict pseudoconvexity of f, $f(y) > f(x)$ for all $y \in s$

Hence, x is the global minimum of f over s

Let x' be another global minimum of f over s.

By assumption, $\nabla f(x)^t (x' - x) \geq 0$

By the strict pseudoconvexity of f, $f(x') > f(x)$

This is a contradiction

Hence x is the unique global minimum of f over s.

Corollary 2.3. Let f be strictly pseudoconvex on s.

If $\nabla f(x)=0$, then x is the unique global minimum of f over s.

(proof) Since $\nabla f(x)=0$, $\nabla f(x)^t(y-x)=0$ for all $y \in s$

By Theorem 2.2, x is the unique global minimum of f over s

Theorem 2.4. Let s be an open convex set in En and $f: s \rightarrow E_1$ be differentiable. Let f be quasiconvex on s.

Suppose that $\nabla f(x)^t(y-x)=0$, $y \neq x$ and $x, y \in s$ implies $f(x) \neq f(y)$

Then f is strictly pseudoconvex on s if and only if $\nabla f(x)=0$ implies that x is the unique global minimum of f over s.

(proof) ($\Rightarrow$) By corollary 2.3, it is clear.

($\Leftarrow$) Suppose that f is not strictly pseudoconvex on s.

Then, there exists a pair $x, y \in s$ $(x \neq y)$ s.t. $f(y) \leq f(x)$ and $\nabla f(x)^t(y-x) \geq 0$ (2.1)

Since $f(y) \leq f(x)$, x is not the unique minimum of f over s. By assumption, $\nabla f(x) \neq 0$ (2.2)

Since f is quasiconvex on s and $f(y) \leq f(x)$, $\nabla f(x)^t(y-x) \leq 0$ (2.3)

From (2.1) and (2.3), $\nabla f(x)^t(y-x)=0$

By assumption, $f(y) \neq f(x)$

On the other hand, f being differentiable on s, f is continuous on s.

Since $f(y) < f(x)$, there exists a real number $\theta > 0$ such that $f[y+\theta \nabla f(x)] < f(x)$

By the quasiconvexity of f, $\nabla f(x)^t[y+\theta \nabla f(x)-x] \leq 0$

$$
\begin{aligned}
0 &\geq \nabla f(x)^t[y+\theta \nabla f(x)-x] \\
&= \nabla f(x)^t(y-x)+\theta \nabla f(x)^t \nabla f(x) \\
&= \theta \nabla f(x)^t \nabla f(x)
\end{aligned}
$$

Since $\theta > 0$, $0 \geq \nabla f(x)^t \nabla f(x) = \| \nabla f(x) \|$

Hence, $\nabla f(x)=0$

This contradicts to (2.2)

Hence f is strictly pseudoconvex on s.

References

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