解의 漸近擧動

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Asymptotic Behavior of Solutions

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(Ⅰ) Abstract

First of all, Under the assusmption that a solution X(t) is bounded and approaches a closed set Ω, it will be shown that the positive limit set of X(t) is composed of solution of some system defined on Ω which is related to the unperturbed system.

We present some conditions which ensure that the solution $Y(x)$ of the ordinary differential equation

$$Y'(x)=A(x)Y(x),\quad Y(x_0)=I,\ \text{where}\ x_0\leq x<\infty\ \text{and}\ A(x),\ Y(x)$$

are n×n complex matrix-valued functions with $A(x)$ continuous, has a nonsingular limit as $x\to\infty$.

(Ⅱ) The theorem

(A) For a system defined on a set D

$$X'=f(x),\quad x\varepsilon D \tag{1}$$

and for subset M of D, M is said to be a semi-invariant set of (1), if for each point of M there exists at least one solution of (1) which remains in M for all future time. consider a system

$$X'=f(t,x)+g(t,x) \tag{2}$$

Let Q be an open set in R^n and suppose that $f(t,x)$, $g(t,x)$ are continuous on $I\times Q$.

Moreover Suppose that if $X(t)$ is continuous and bounded on $[t_0,\infty)$, that is, for some compact set

$Q^*\subset Q$, $X(t)\epsilon Q^*$ for all $t\epsilon[t_0,\infty)$, then we have

$$\int_{t_0}^{\infty}|g(s,X(s))|ds<\infty \tag{3}$$

Let $X(t,t_0,x_0)$ be a solution of (2) through (t_0,x_0) and let Γ^+ be a set in R^n such that

$$\Gamma^+=\bigcap_{t_0\leq s<\tau}\ \bigcup_{s\leq t}\times(t,t_0,x_0)$$

where the interval $[t_0,\tau)$ is the maximal interval of the solution $X(t,t_0,x_0)$.

Then Γ^+ is a closed subset in Q, and if $X(t,t_0x_0)$ is bounded, Γ^+ is non-empty and compact.

Let Ω be a closed set in the space Q, suppose that a solution $X(t)$ approaches Ω as $t\to\infty$. Then the positive limit set Γ^+ of $X(t)$ is contained in Ω.

Now we shall make the following assumption for the system(2).

Let Ω be a nonempty closed set in the space Q and suppose that $f(t,x)$ of(2) satisfies the following conditions:

i) $f(t,x)$ tends to a function $h(X)$ for $X\varepsilon\Omega$ as $t\to\infty$ and on any compact set in Ω this convergence is uniform.

ii) corresponding to each $\varepsilon>0$ and each $Y\varepsilon\Omega$, there exists a $\delta(\varepsilon,y)>0$ and a $T(\varepsilon,y>)0$ such that if $|x-y|<\delta(\varepsilon)$ and $t\leq T(\varepsilon,y)$, We have

$$|f(t,x)-(t,y)|<\varepsilon$$

if $f(t,x)$ satisfies condition (ii), for $y\varepsilon\Omega_1$, where Ω_1, is a compact set in Ω, We can choose δ and T which are in dependent of y and depend only on Ω_1.

Theorem 1

Suppose that a solution $X(t,t_0,x_0)$ of (2) is bounded and approaches a closed set Ω in the space Q.

if $f(t.x)$ satisfies conditions i) and ii), then the positive limit set Γ^+ of $X(t,t_0,x_0)$ is a semi-invariant set contained in Ω of the equation.

$$X'h(x),\quad X\varepsilon\Omega \qquad (4)$$

proof)

$|X(t_k)-w|<\delta$ for sufficiently large k.

Therefore, there exists a solution $\phi_k(t)$ defined on $0\leq t\leq\lambda$ of System.

$X'=h^*(x)$, $X\varepsilon R^n$ through $(0,w)$ such that for a given $\varepsilon>0$, $|X(t+t_k)-\phi_k(t)|<\varepsilon$ for $t\varepsilon[0,\lambda]$, since $X(t)\to\Gamma^+$ as $t\to\infty$, if k is sufficiently large, $\phi_k(t)\varepsilon N(2\varepsilon,1)$ for $t\varepsilon[0,\lambda]$, and also.

$$\phi_k(t)=w+\int_0^t h^*(\phi_k(s))ds \text{ for } t\varepsilon[0,X].$$

Thus, for a sequence $\{\varepsilon_k\}$ approaching zero as $K\to\infty$, there exist solution $\phi_k(t)$ of $X'=h^*(x)$, $X\varepsilon R^n$ such That

$$\begin{cases}\phi_k(t)=w+\int_0^t h^*(\phi_k(s))ds\\ \phi_k(t)\varepsilon N(\varepsilon_k,\Gamma^+)\end{cases} \qquad (5)$$

for $t\varepsilon[0,\lambda]$. Since $\{\phi_k(t)\}$ is uniformly bounded and equicontinuous, it has a uniformly convergent subsequence. Let $\phi(t)$ be its limit function. Then, by (5),

$$\phi(t)=w+\int_0^t h^*(\phi(s))ds,\ \phi(t)\varepsilon\Gamma^+ \text{ for } t\varepsilon[0,\lambda].$$

since $\Gamma^+\subset\Omega_1$, $h^*(\phi(t))=h(\phi))$, which implies that.

$\phi(t)=w+\int_0^t h(\phi(s))ds$ for $t\varepsilon[0,\lambda]$, that is, $\phi(t)$ is a solution of system (4) through $(0,w)$

and remain in Γ^+. Since λ is arbitrary, We can find a solution of (4) defined on I which passes through $(0, w)$ and remains in Γ^+.

This proves that Γ^+ is a semi invariant set of (4).

(B) consider the initial value problem

$$Y'(x)=A(x)Y(x),\quad x_0\leq x<\infty$$
$$Y(x_0)=I \qquad (6)$$

Here $A(x)$ and $Y(x)$ are $n\times n$ complex matrix-x_0 valued function, and $A(x)$ is assumed to be continuous. The purpose of the paper is to give some sufficient conditious for lim $x\to\infty Y(x)$ to exist and be invertible it is well knorwn[1] that one sufficient condition is

$A(x)\varepsilon L^1<0,\infty)$, ie, $\int_{x_0}^{x}\|A(x)\|dx<$ Where$\|\ \|$is the norm (any of the equivalent ones) on the space of n×n complex matrices.

On the other hand, if $A(x)$ is a commutative family, the solution to (6) is

$$Y(x)=\exp\left(\int_{x_0}^{x}A(s)\mathrm{d}s\right)$$

and so $Y(x)$ has an invertible limit at ∞ provided the improper integral

lim $x\to\infty\int_{x_0}^{x}A(s)\mathrm{d}s=\int_{x_0}^{x}A(d)\mathrm{d}s$ exists.

Theorem 2

Suppose $A(x)=B(x)+C(x)$, where $B(x)$, $C(x)$ are continous, $C(x)\varepsilon L^1(x.,\infty)$, $H(x)=\int_x^\infty B(s)\mathrm{d}s$ exist as an improper Riemann integral, and $H(x)$ $B(x)\varepsilon L^1(x_0,\infty)$. Then lim$x\to\infty Y(x)$ exist and is invertible.

proof of the theerem. Let $Y(x)$ be the solution of (6), and let

$$Z(x)=(1+H(x))Y(x)$$

Since the first factor has the limit I as $x\to\infty$, it suffices to show that $Z(x)$ has an invertible limit at ∞. Now for $a<x<\infty$ and a sufficiently large we have

$$\begin{aligned}Z(x)&=(A(x)+H(x)A(x))Y(x)-B(x)Y(x)\\&=(C(x)+H(x)C(x)+H(x)B(x))Y(x)\\&=R(x)Z(x),\quad R(x)\varepsilon L^1(a,\infty)\end{aligned}$$

This completes the proof of theorem.

Theorem 3

Suppose $A(x)=B_0(x)+C_0(x)$, $C_0(x)\varepsilon L^1(x_0,\infty)$

$$\begin{aligned}H_1(x)&=\int_x^\infty B_0(s)\mathrm{d}s=B_1(x)+C_1(x),\\&\quad C_1(x)A(x)\varepsilon L^1(x_0,\infty)\\H_2(x)&=\int_x^\infty B_1(s)A(s)\mathrm{d}s=B_2(x)+C_2(x)\\&\quad C_2(x)A(x)\varepsilon L^1(x_0,\infty)\\&\vdots\\H_{n-1}(x)&=\int_x^\infty B_{n-2}(s)A(s)\mathrm{d}s=B_{n-1}(x)+C_{n-1}(x)\\&\quad C_{n-1}(x)A(x)\varepsilon L^1(x_0,\infty)\\H_n(x)&=\int_x^\infty B_{n-1}(s)A(s)\mathrm{d}s\\&\quad H_{n-1}(x)A(x)\varepsilon L^1(t_0,\infty)\end{aligned}$$

Then lim $x\to\infty Y(x)$ exists and is invertible (all matrix-valued functions are assumed continous, and $\int_x^\infty$ integrals are improper Riemann integrals) proof) The proof is analogous to that of theorem 2. Let $Y(x)$ be the solution of (6) and

$$Z(x) = (I + H_1(x) + \cdots\cdots + H_n(x))Y(x)$$

The first factor has limit I at ∞ A_o so We need only show that $Z(x)$ has an invertible limit at ∞. For $a < x < \infty$ and a sufficiently large We have

$$\begin{aligned} Z'(x) &= (A(x) + H_1(x)A(x) + \cdots\cdots + H_n(x)A(x))Y(x) \\ &\quad + (-B_o(x) - B_1(x)A(x) - \cdots\cdots - B_{n-1}(x)A(x))Y(x) \\ &= (C_o(x) - C_1(x)A(x) + \cdots\cdots C_{n-1}(x)A(x) \\ &\quad + H_n(x)A(x)Y(x) \\ &= R(x)Z(x), \quad R(x) \varepsilon L^1(a, \infty) \end{aligned}$$

Which proves theorem 3.

Ⅲ. References

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