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The Existence and Uniqueness of Solution to the Volterra Integrodifferential Equation of Parabolic Type

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1. Introduction

We consider nonlinear heat flow in a homogeneous bar of unit length of material with memory with the temperature $u = u(t,x)$ maintained at zero at $x = 0$ and $x = 1$ by the similar method as [2]. We shall assume that the history of u is prescribed for $t \geq 0$ and $0 \leq x \leq 1$. The equation satisfied by u in such a material is derived from assumptions that the internal energy ε and the heat flux q are functionals of u and of the gradient of u, respectively. According to the theory developed by Nunziato [13] for heat flow in materials of fading memory type the functionals ε and q are taken respectively as

(1.1) $$\varepsilon(t,x) = \alpha u(t,x) + \int_0^t \beta(t-s)u(s,x)ds,$$

(1.2) $$q(t,x) = -\kappa u_x(t,x) - \int_0^t a(t-s)\sigma(u_x(s,x))ds$$

where $\alpha > 0, \kappa > 0$ are given constants and $\beta, a : [0,\infty) \longrightarrow \mathbf{R}$ are given sufficiently smooth functions (called the internal energy and heat flux relaxation functions, respectively). In the physical literature β and a are usually assumed to be decaying exponentials with positive coefficients. The real function $\sigma : \mathbf{R} \longrightarrow \mathbf{R}$ in (1.2) will be assumed to $\sigma \in \mathbf{C}^1(\mathbf{R}), \sigma(0) = 0$. It should be noted that the $\sigma(r) \equiv r$ gives rise to the linear model derived in Nunziato [13], and that (1.2) is one reasonable generalization of the heat flux for nonlinear heat flow in one space dimension.

If $f = f(t,x) \in \mathbf{C}^1([0,\infty); \mathbf{L}^1(0,1))$ represents the external heat supply added to the rod for $t \geq 0$ and $0 \leq x \leq 1$, and if $u(0,x) = u_0(x), 0 \leq x \leq 1$ is the given initial temperature distribution, the law of balance of heat ($\varepsilon_t = -divq + f$) shows that in one space dimension the temperature u satisfies the initial boundary value problem

$$
\begin{aligned}
(1.3) \qquad & \frac{\partial}{\partial t}\{\alpha u(t,x) + \int_0^t \beta(t-s)u(s,x)ds\} \\
& - \kappa u_{xx}(t,x) - \int_0^t a(t-s)(\sigma(u_x(s,x)))_x ds \\
& = f(t,x),
\end{aligned}
$$

$$u(0,x) = u_0(x) \ \ (0 \leq x \leq 1), \ \ u(t,0) = u(t,1) \equiv 0 \ \ (t \geq 0),$$

where subscripts denote differentiation with respect to x.

In the case $\beta(t) \equiv 0, \alpha = \kappa = 1$, in Section 3 we consider the initial boundary value problem to the following equation:

$$
\begin{aligned}
(1.4) \qquad & u_t(t,x) - u_{xx}(t,x) - \int_0^t a(t-s)(\sigma(u_x(s,x)))_x ds \\
& = f(t,x), \quad 0 \leq x \leq 1, \ \ t \geq 0,
\end{aligned}
$$

$$(1.5) \qquad u(t,0) = u(t,1) = 0, \ \ t \geq 0,$$

(1.6) $$u(0,x) = u_0(x), \quad 0 \le x \le 1.$$

2. Preliminaries

Here, all functions may be real value. We denote by $\mathbf{L}^2(0,1)$ consist of functions f on some interval [0,1] such that $|f(t)|^2$ is Lebesgue integrable on this interval. The space $\mathbf{L}^2(0,1)$ is Hilbert space with the inner product

$$(f,g) = \int_0^1 f(x)g(x)dx,$$

and norm is given by

$$|f| = \left(\int_0^1 f(x)^2 dx\right)^{1/2}.$$

$\mathbf{H}^1(0,1)$ denotes the set of all functions whose u is absolutely continuous on [0,1] and whose derivatives up to degree 1 belong to $\mathbf{L}^2(0,1)$. $\mathbf{H}^1(0,1)$ is a Hilbert space with inner product and norm by

$$((u,v))_1 = (u',v') + (u,v),$$

$$\|u\|_1 = (|u'|^2 + |u|^2)^{1/2},$$

respectively, for all $u, v \in \mathbf{H}^1(0,1)$.

Denotes

$$\mathbf{H}_0^1(0,1) = \{u \in \mathbf{H}^1(0,1) \mid u(0) = u(1) = 0\},$$

$$\mathbf{H}_0^1(0,1) \subset \mathbf{H}^1(0,1).$$

Hence $\mathbf{H}_0^1(0,1)$ is a Hilbert space with inner product and norm by

$$((u,v)) = \int_0^1 u'(x)v'(x)dx,$$

$$\|u\| = \left(\int_0^1 u'(x)^2 dx\right)^{1/2},$$

respectively, for all $u, v \in \mathbf{H}_0^1(0,1)$. Denotes $\mathbf{H}^2(0,1) = \{u \in \mathbf{H}^1(0,1) \mid u'$; absolutely continuous on [0,1], $u'' \in \mathbf{L}^2(0,1)\}$. Hence $\mathbf{H}^2(0,1)$ is a Hilbert space with inner product and norm

$$((u,v))_2 = (u'', v'') + (u', v') + (u, v),$$

$$\|u\|_2 = (|u''|^2 + |u'|^2 + |u|^2)^{1/2},$$

respectively, for all $u, v \in \mathbf{H}^2(0,1)$. $\mathbf{H}^{-1}(0,1) = \mathbf{H}_0^1(0,1)^*$ denotes the set of all functions whose derivatives up to degree 1 in distribution sense belong to $\mathbf{L}^2(0,1)$. i.e.,

$$\mathbf{H}^{-1}(0,1) = \{\frac{df}{dx} \mid f \in \mathbf{L}^2(0,1)\}$$

where $\mathbf{H}_0^1(0,1)^*$ is dual space of $\mathbf{H}_0^1(0,1)$. By the definition, if $u \in \mathbf{H}^{-1}(0,1)$ then there exist $f \in \mathbf{L}^2(0,1)$ such that $u = f'$ but f don't exist uniquely. As a matter of fact, let $u = f'$ and c is a constant then $f + c \in \mathbf{L}^2(0,1)$, $u = (f+c)'$.

Conversely, suppose that $f,\ g \in \mathbf{L}^2(0,1)$ and $f' = g' = u$, then $(f-g)' = 0$. Hence f is constant function. If we select f such that $\int_0^1 f(x)dx = 0$, then f is uniquely determined by u. Let

$$u,\ v \in \mathbf{H}^{-1}(0,1), \quad u = f', \quad v = g', \quad \int_0^1 f(x)dx = \int_0^1 g(x)dx = 0$$

and

$$((u,v))_* = (f,g), \quad \|u\|_* = |f|,$$

then $\mathbf{H}^{-1}(0,1)$ becomes a Hilbert space with inner product $((\cdot,\cdot))_*$ and norm $\|\cdot\|_*$. Therefore,

$$\mathbf{H}_0^1(0,1) \subset \mathbf{L}^2(0,1) \subset \mathbf{H}^{-1}(0,1).$$

3. Assumptions and main theorem

By the definition,

$$\mathbf{H}^2(0,1) \cap \mathbf{H}_0^1(0,1) = \{u \in \mathbf{H}^2(0,1) \mid u(0) = u(1) = 0\}.$$

Let A be the operator defined by

(3.1) $$D(A) = \text{domain of } A = \{u|u \in \mathbf{H}^2(0,1) \cap \mathbf{H}_0^1(0,1)\},$$

(3.2) $$Au = -\Delta u \ (\Delta = \text{Laplacian}), \qquad u \in D(A).$$

Then A is positive definite self–adjoint operator. Suppose that $\sigma(r)$ is defined continuously differentiable on $\mathbf{R}$, the derivative $\sigma'(r)$ is bounded. That is, there exists $M > 0$ such that

(3.3) $$|\sigma'(r)| \leq M$$

for $-\infty < r < \infty$. Obviously $\sigma(r)$ satisfies uniformly Lipschitz condition. In other words,

(3.4) $$|\sigma(r) - \sigma(s)| \leq M|r - s|$$

for $r, s \in \mathbf{R}$. For $u \in D(A)$ and by (3.4), the following inequality holds.

$$|\sigma(u'(x)) - \sigma(u'(y))| \leq M|u'(x) - u'(y)|.$$

Hence $\sigma(u'(x))$ is absolutely continuous. For $u \in D(A)$, the following equality holds

$$(\sigma(u'(x)))' = \sigma'(u'(x))u''(x). \tag{3.5}$$

Defining $g(u(x)) = (\sigma(u'(x)))'$ then $g(u)(x) = \sigma'(u'(x))u''(x)$. By virtue of (3.3), for each $u \in D(A)$, it follows that

$$g(u) \in \mathbf{L}^2(0,1), \ \ |g(u)| \le M|u''|.$$

We need the following assumptions:

(i) $a(t)$ is Hölder continuous on $[0,\infty)$ with exponent ρ, that is, there exists a constant $c > 0$ and $0 < \rho \le 1$ such that

$$|a(t) - a(s)| \le c|t-s|^{\rho}$$

for all $t, s \in [0,\infty)$.

(ii) $u_0 \in D(A)$.

(iii) $f(t) \in \mathbf{C}^1([0,\infty); \mathbf{L}^2(0,1))$.

We write the mixed problem (1.2) – (1.4) as a formulation in $\mathbf{L}^2(0,1)$,

$$\frac{du}{dt}(t) + Au(t) + \int_0^t a(t-s)g(u(s))ds = f(t), \tag{3.6}$$

$$u(0) = u_0. \tag{3.7}$$

Theorem 3.1. Assume that (i)–(iii) hold. Let $u(t), f(t)$ be a $\mathbf{L}^2(0,1)$ – valued functions of t, respectively and

$$u(t) \in D(A), \ \frac{du}{dt}(t) = \lim_{h\to 0} \frac{u(t+h)-u(t)}{h}$$

be exist in the $\mathbf{L}^2(0,1)$-topology. Then there exist a unique solution $u(t)$ of (3.6), (3.7).

We can solve the problem (3.6), (3.7) by the following method. Setting $q(v,u) = \sigma'(v')u''$ for all $v \in \mathbf{H}_0^1(0,1)$ and for all $u \in D(A)$. Obviously, $q(u,u) = g(u)$.

(a) Let $v(t)$ be a $\mathbf{H}_0^1(0,1)$–valued continuous function such that $v(0) = u_0$. We solve that the following initial value problem:

$$\frac{du}{dt}(t) + Au(t) + \int_0^t a(t-s)q(v(s),u(s))ds = f(t), \tag{3.8}$$

$$u(0) = u_0. \tag{3.9}$$

(b) Since the solution of Eqs.(3.8),(3.9) is exist and unique, the mapping θ defined from $\mathbf{H}_0^1(0,1)$ to $D(A)$ i.e.,$\theta v = u$. Therefore there exists a fixed point of θ, that is, the fixed point is a solution of Eqs.(3.6),(3.7).

Proof of the Existence [7, 8]. (a) Obviously, $q(v,u)$ satisfies uniformly Lipschitz condition that the following inequality : there exists $M > 0$ such that

$$|q(v,u_1) - q(v,u_2)| \leq M\|u_1 - u_2\|$$

for all $v \in \mathbf{H}_0^1(0,1)$ and $u_1, u_2 \in D(A)$. Since A is a positive definite self–adjoint operator, we set

$$A = \int_0^\infty \lambda dE(\lambda) \ ; \ \text{(real) spectral resolution of operator } A,$$

where E is a real spectral measure.

We are defined $\exp(-tA) = \int_0^\infty \exp(-\lambda t)E(\lambda)$ for all $t \geq 0$. It is known that, $-A$ generates an analytic semigroup in $\mathbf{L}^p(0,1)\,(1 \leq p < \infty)$. For each $v \in D(A)$ we define

$$(K_v u)(t) = \exp(-tA)u_0$$
$$+ \int_0^t \exp(-(t-s)A)\{f(s) - \int_0^s a(s-r)q(v(r),u(r))dr\}ds.$$

We show that for t_0 sufficiently small, K_v is a contraction. By (3.3), there exists a constant c $(0 < c < 1)$ that satisfy a following inequality:

$$\|(K_v u_1)(t) - (K_v u_2)(t)\| \leq c\|u_1(t) - u_2(t)\|$$

for all $0 \leq t \leq t_0$. It follows that K_v is a contraction mapping of $D(A)$ into $\mathbf{H}^2(0,1)$. Hence there is a unique fixed point u of K_v in $\mathbf{H}^2(0,1)$ and $u(t)$ is a local solution of (3.8),(3.9). Then by previous arguments there exists a unique solution $u(t)$ of (3.8),(3.9) on $[0,\infty)$.

(b) Let t_1, L, η are positive number and $\eta < 1$. Define

$$S = \{v \in \mathbf{H}_0^1(0,1) \mid \ \|v(t) - v(s)\| \leq L|t-s|^\eta, v(0) = u_0 \ \text{ in } [0,t_1]\}.$$

For t_1 sufficiently small, then $\theta : S \to S$ is continuously compact mapping for some L, η. Since S is closed, bounded, and convex, it follows from Schauder's fixed point theorem there is a unique fixed point that satisfies (3.6),(3.7) on $[0,t_1]$. By the continuity, there exists a solution of (3.6),(3.7) on $[0,\infty)$.

Proof of the Uniqueness. In the previous proof we have only the existence of solution of Eqs.(1.2) – (1.4) in $\mathbf{L}^2(0,1)$. It remains to prove the uniqueness of (1.2) – (1.4). For this purpose we need some elements of functional analysis. Putting

$$\widetilde{A}u = -u'' \quad \text{for all } u \in \mathbf{H}_0^1(0.1).$$

Since $-u'' = (-u')'$, $-u' \in \mathbf{L}^2(0,1)$ hold then

$$\widetilde{A}u \in \mathbf{H}^{-1}(0,1), \quad \int_0^1 -u'(x)dx = -u(1) + u(0) = 0.$$

Hence, if for $v \in \mathbf{H}_0^1(0,1)$, the function g satisfies

$$g' = v, \quad \int_0^1 g(x)dx = 0,$$

then by integration by parts and $u(0) = u(1) = 0$.

$$((\widetilde{A}u, v))_* = (-u', g) = (u, g') = (u, v),$$

$$((u, \widetilde{A}v))_* = ((\widetilde{A}v, u))_* = (v, u) = (u, v) = ((\widetilde{A}u, v))_*.$$

Therefore, the operator A is a symmetric and a positive definite self–adjoint on $\mathbf{H}^{-1}(0,1)$. Defined the linear operator $\widetilde{A} : \mathbf{H}_0^1(0,1) \to \mathbf{H}^{-1}(0,1)$ by

$$D(A) = \{u \in \mathbf{H}_0^1(0,1) \,|\, \widetilde{A}u \in \mathbf{L}^2(0,1)\}, \tag{3.10}$$

$$\widetilde{A}u = Au \quad \text{for all } u \in D(A). \tag{3.11}$$

Since we may replace $\sigma(r)$ by $\sigma(r)$ + constants without altering equation (1.4), we may assume that $\sigma(0) = 0$, by (3.4) for all $u \in \mathbf{H}_0^1(0,1)$ then

$$|\sigma(u'(x))| = |\sigma(u'(x)) - \sigma(0)| \leq M|u'(x)|.$$

Hence

$$\sigma(u') \in \mathbf{L}^2(0,1), \quad (\sigma(u'))' \in \mathbf{H}^{-1}(0,1).$$

And if we put $\widetilde{g}(u) = (\sigma(u'))'$ then $\widetilde{g}$ is a mapping from $\mathbf{H}_0^1(0,1)$ to $\mathbf{H}^{-1}(0,1)$. In particularly, if $u \in D(A)$, then $\widetilde{g}(u) = g(u)$. By the above argument, we consider the mixed problem (1.4) – (1.6) as a formulation in $\mathbf{H}^{-1}(0,1)$:

$$\text{(3.12)} \qquad \frac{du}{dt}(t) + \widetilde{A}u(t) + \int_0^t a(t-s)\widetilde{g}(u(s))ds = f(t),$$

$$\text{(3.13).} \qquad u(0) = u_0$$

Obviously, if the solution of initial value problem (3.6) – (3.7) exists then that of (3.12) – (3.13) exists, and the solution of initial value problem (3.12) – (3.13) is unique then that of (3.6) – (3.7) is unique.

We prove that the solution (3.12) – (3.13) is unique. In general, if $f \in \mathbf{L}^2(0,1)$ then $\|f'\|_* \leq |f|$. By (3.4), we have

$$\begin{aligned} \|\tilde{g}(u) - \tilde{g}(v)\|_* &= \|(\sigma(u'))' - (\sigma(v'))'\|_* \\ &\leq |\sigma(u') - \sigma(v')| \\ &\leq M|u' - v'| = M\|u - v\|. \end{aligned}$$

Hence $\widetilde{g}$ satisfies a uniform Lipschitz condition. Therefore the uniqueness of solution of (3.12), (3.13) follows from (3.8), (3.9).

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