位相的 擴張의 C*-埋藏이 될 特性表示에 關한 硏究

金 章 郁

Characterization of C*-embedding of Topological Extensions

Chang Wook Kim

〈目 次〉

Ⅰ. Introduction

Let X be a Hausdorff space. In this note, we shall discuss the necessary and sufficient conditions of the C^*-embedding of the topological extensions of X.

Hausdorff 空間X가 X의 어떤 位相的 擴張의 C^*-埋藏이 될 必要 充分條件에 대해 고찰한다.

Ⅱ. Main Results

Definition 1. A function $f: X \to Y$ is θ-continuous if for each $p \in X$ and open set U containing $f(p)$, there is open set V containing p such $f(Cl(V) \subset Cl(f(U))$. And f is θ-homeomorphism if f is θ-continuous, bijective and f^{-1} is θ-continuous.

Definition 2[3]. The semiregularization of X, denoted by X, is the space whose set is the set of X with the topology generated by open basis $\{Int(Cl(U)) | U \text{ is open in } X\}$. And X is semiregular if the topology of X is generated by $\{Int(Cl(U)) | U \text{ is open in } X\}$.

Definition 3. Y is an extension of X if X is a dense subspace of Y. And Y is a P extension of X if Y is an extension of X and Y satisfies the topological property P.

Definition 4. An extension Y of X is θ-isomorphic to an extension Z of X if there exists a θ-homeomorphism f: $X \to Y$ such that $f(x)=x$ for each $x \subset X$.

Definition 5[3]. A Hausdorff space X is H-closed if every open cover of X has a finite subfamily whose union is dense.

Definition 6[2]. Let X be a Hausdorff space and T a Hausdorff extension of X. For $p \in T$, define $\theta^p = \{U \cap X \mid U$ is open in T and $p \in U\}$. The extension T of X has the simple extension topology if $\{U \cap \{p\} \mid U$ is open in X, $p \in$ T and $U \in \theta^p\}$ is an open basis for the topology of T.

Definition 7[4]. The Hausdorff extension T of X is called the Katětov extension of X and denoted kX if $\{\theta^p \mid p \in T \setminus X\}$ and T has the simple extension topology.

Definition 8[4]. Let $C^*(X)(C^*(Y))$ denotes the set of all continuous functions on $X(Y)$ and let Y is an extension of X. X is C^*-embedded in Yif for every g in $C^*(X)$, there is a g' in $C^*(Y)$ such that g' is a continuous extension of g.

Proposition 1[5]. For each H-closed extension hX of X, there is a unique continuous function f: $kX \to hX$ such that $f(x)=x$ for $x \in X$ and $f(kX \setminus X) = hX \setminus X$.

Definition 9. The function f in Proposition 1 is called the Katětov function of hX.

Remark. Each H-closed extension hX of X determines a partition
$P\{hX\} = \{f^{-1}(p) \mid p \in hX\}$ of kX by the Katětov function. In fact
$P(hX) = \{\{x\} \mid x \in X\} \cup \{f^{-1}(p) \mid p \in hX \setminus X\}$.

Proposition 2[7]. Let hX be an H-closed extension of X, Then there is an H-closed extension $h'X$ such that$P(h'X)=P(hX)$, the Katětov function f: $kX \to h'X$ is a quotient map and $h'X$ and hX are θ-isomorphic extension of X.

Remark. For $p \in kX \setminus X$, let $A(p) = \cap \{f^{-1}f(p) \setminus X \mid f \in C^*(kX)\}$and let $P^*(X) = \{A(p) \mid p \in kX \setminus X\} \cup \{\{x\} \mid x \in X\}$, $P^*(X)$ is a partition of kX.

Definition 10. Let Q be a family of pairwise disjoint subset of kX, Q refines $P^*(X)$ means for each $A \in Q$, there is $B \in P^*(X)$ such that $A \subseteq B$.

Proposition 3. A space X is C^*-embedded in the H-closed extension hX of X if and only if $P(hX)$ refines $P^*(X)$.

Proof. Suppose that X is C^*-embedded in hX. Since hX is an H-closed extension of X, by Proposition 1, there is a Katětov function f: $kX \to hX$ such that $f(x)=x$ for all $x \in X$ and $f(kX \setminus X)=hX \setminus X$. And since $P(hX)=\{\{x\} \mid x \subset X\} \cup \{f^{-1}(p) \mid p \in hX \setminus X\}$, to prove our assertion, it is sufficient to show that for $f(q)=p \in hX \setminus X$, $f^{-1}(p) \subseteq A(q)$. For this purpose, let $r \in f^{-1}(p)$ and $g \in C^*(kX)$, then $g|_X$ is in $C^*(X)$ and hence there is k is in $C^*(hX)$ such that k is a continuous extension of $g|_X$ and $q \in f^{-1}(p)$ and $f(r)=p$. Since $g|_X=k \cdot f$. Thus $g(r)=k(p)=g(q)$ and hence $r \in A(q)$. Thus $P(hX)$ refines $P^*(X)$. Conversely, suppose $P(hX)$ refines $P^*(X)$. By Proposition 2, there is an H-closed extension $h'X$ of X such that $P(h'X)=P(hX)$, the Katětov function f: $kX \to h'X$ is a quotient map and $h'X$ and hX are θ-isomorphic extension of X. Let $g \in C^*(X)$ and g' be the continuous extension of g to kX. Since $P(h'X)=P(hX)$ refines $P^*(X)$, $f^{-1}(p) \subseteq A(q)$ for $f(q)=p$ $h'X$ X. Hence g' is constant on $f^{-1}(p)$. Define the function $g'' \in C^*(h'X)$ by $g''(p)=g'(f^{-1}(p))$. Since $g'=g'' \cdot f$ and since $h'X$ has the quotient topology induced by f, g'' is continuous. Since $h'X$ and hX are θ-isomorphic extension of X, there is a θ-homeomorphism k: $hX \to h'X$ such that $k(x)=x$ for $x \in X$. Hence k g'' is a θ-continuous extension of g to hX. Since θ-continuous function into a regular space is continuous, k g'' is continuous and k $g'' \in C^*(hX)$. Hence X is C^*-embedded in hX.

Proposition 4. A space X is C^*-embedded in the Hausdorff extension eX of X if and only if $P(eX)$ refines $P^*(X)$.

Proof. Let eX be a Hausdorff extension of X. Then keX is an H-closed extension of X. Let f: $kX \to keX$ be the Katětov function of keX, $P(eX)=\{f^{-1}(p) \mid p \in eX\}$and $M=f^{-1}(eX)$, then $X \subseteq M \subseteq kX$ and $P(eX)$ is a family of pairwise disjoint subsets of kX. Since f is one-to one on $(kX \setminus M) \cup X$, $P(eX)$ refines $P^*(X)$ if and only if $P(keX)$ refines $P^*(X)$. Since eX is C^*-embedded in keX, X is C^*-embedded in eX if and only if X is C^*-embedded in keX. Hence, by Proposition 3, Our assertion is valid.

Corollary 1. Let X be a regular Hausdorff space on which every continuous function is constant. Then X is C^*-embedded in every H-closed extension of X.

Corollary 2. Let X be Tychonoff. Then the Stone-Cech compactification βX of X has the property that $P(\beta X)=P^*(X)$.

References

1. B. Banaschewski, On the Katětov and Stone-Cech extensions, Canad. Math. Bull. 2(1959), 1—4.
2. ______________, Extnsions of topological spaces, Canad. Math. Bull. 7(1964), 1—22.
3. N. Bourbaki, General Topology, Part 1, Addison-Wesley, 1966.
4. M. Katetov, Uber H-abgeschlossene and bikompake Raume, Casopis Pest. Mat. Pys. 69(1940), 36—49.
5. C.-T. Liu, Absolutely closed spaces, Trans. Amer. Math. Soc. 130(1968), 86—104.
6. J.R. Porter and J.D. Thomas, On H-closed and Minimal Hausdorff spaces, Trans. Amer. Math. Soc. 138(1969), 159—170.
7. __________, and C. Votaw, H-closed extensions, General Topology and its Application, to appear.
8. N.V. Velicko, H-closed topological spaces, Mat. Sb. 70(112)(1966), 98—112. (Amer. Math. Soc. Translation, 78(2)(1968), 103—118.)