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A Study Individual Number Process Under Continuous-Time Markov Chains

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Abstract

In this paper, the individual number of the future has depended not only upon the present individual number but upon the present individual age, considering the stochastic process model if individual number when the life span of each individual number and the individual age as a set, this becomes a Markovian. Therefore, in this paper the individual is treated as invariable, without depending upon the whole record of each individual since its birth. As a result, suppose $\{N(t),\ t > 0\}$ be a counting process and also suppose Z_n denote the life span between the (n-1)st and the nth event of this process, $(n \geq 1)$: that is, when the first individual is established at $n=1$(time 0), the Z_n at time the nth individual breaks down. Random walk Z_n is

$Z_n = X_1 + X_2 + \cdots\cdots + X_n,\ Z_0 = 0$

So, fixed time t, the stochastic model is made up as follows;

A) Recurrence(Regeneration) number between(0,t)

$N_t = \max\{n ; Z_n \leq t\}$

B) Forward recurrence time(Excess life)

$T_t^+ = Z_{N_t+1} - t$

C) Backward recurrence time(Current life)

$T_t^- = t - Z_{N_t}$

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1. Introduction

In making a stochastic model for a real phenomenon, which is always necessary to make certain simplifing assumption so as to under the probability tractable. On the other hand, however, we can not make too many simplifing assumptions, for then our conclusions, obtained from the stochastic model, would not be applicable to the real world phenomenon. Thus, in short, we must make enough simplifing assumption to enable us to handle the probability but not so many that the stochastic model no longer resembles the real world phenomenon.

Therefore, in this paper a class of probability models that has a wide variaty of applications in the real world is using poission process and exponential distribution as random variables most frequently used simplifing assumption.

That is, by obtaining differential equation as to backward equation and forward equation, practical examples about birth and death process are to be applied to real life.

2. Differential Equations for Backward and Forward Equation.

Consider a system whose state at any time is represented by the number of people in the system at that time. Suppose that whenever there are in people in the system, then (i) new arrivals enter the system at an exponential rate λ_n and (ii) people leave the system at an exponential rate μ_n. That is, whenever there are in persons in the system, then the time until the next arrivals is exponentially distributed with mean $1/\lambda_n$ and is independent of the time until the next departure which is itself exponentially distributed with mean $1/\lambda_n$ thus, a birth and death process is a continuous time Markov chain with states $\{0,1,2,\cdots\}$ for which transitions from state n may go only to either state n-1 or state n+1.

The relation between the birth and death rates and the state transition rates and probabilities are

$\nu_0 = \lambda_0$

$\nu_i = \lambda_i + \mu_i,\ i > 0$

$P_{oi} = 1$

$P_{i\,i+1} + 1 = \frac{\lambda_i}{\lambda_i + \mu_i},\ i > 0$

$P_{i\,i+1} - 1 = \frac{\mu_i}{\lambda_i + \mu_i},\ i > 0$

The preceding follows, since when there are i in the system, then the next state will be i+1 if a birth occurs before a death, and the probability that an exponential random variable with rate λ_i will occur earlier then an independe -nt exponential with rate μ_i is $\lambda_i/\lambda_i+\mu_i$

(and so, $P_{i\,i+1} = \frac{\lambda}{\lambda_i + \mu_i}$)

and the time until either occurs is exponentially distributed with rate $\lambda_i+\mu_i$

(and so, $\nu_i = \lambda_i + \mu_i$)

Let $P_{ij}(t) = P\{x(t+s) = j \mid x(s) = i\}$

$P_{ij}(t)$ represents the probability that a Markov process presently in state i will be in state j time later. We shall attempt to derive a set of differential equations for these transition probabilities $P_i(t)$. However, first we will need the following two lemmas.

Lemma 1.

a) $\lim_{h \to o} = \frac{1-P_{ii}(h)}{h} = \nu_i$

b) $\lim_{h \to o} = \frac{P_{ij}(h)}{h} = \nu_i P_{ij},\ (i \neq j)$

Proof: we first note that since the amount of time until a transition occur is exponentially distributed, it follows that the probability of two or more transitions in a time h is o(h). Thus, $1-P_{ii}(h)$, the probability that a process in state i at time o will not be in state i at time h, equals the probability that a transition occur within time h plus something small compared to h. therefore,

$1-P_{ii}(h)=h\nu_i+o(h)$

and part (a) is proved.

To prove the part (b), we note that $P_{i.}(h)$, the probability that the process goes from state i to state j in a time h, equals the probability that a transition occur in this time multiplied by the probability that the transition is into state j, plus something small compared to h, that is,

$P_{i.}(h)=h\nu_{i.}P_{i.}+o(h)$

and part (b) is proved.

Lemma 2.

For all $s\geq 0$, $t\geq 0$

$$P_{ij}(t+s)=\sum_{k=0}^{\infty}P_{ik}(t)\ P_{kj}(s)$$

Proof : In order for the process to go from state i to state j in time t+s, it must be somewhere at time t and thus

$$\begin{aligned}P_{ij}(t+s)&=P\{x(t+s)=j|1\times(0)=j\}\\&=\sum_{k=0}P\{x(t+s)=j,\ x(t)=k|\ x(0)=i\}\\&=\sum_{k=0}^{\infty}P\{x(t+s)=j|x(t)=k,\ x(0)=i\}\ P\{x(t)=k|x(0)=1\}\\&=\sum_{k=0}^{\infty}P\{x(t+s)=j|x(t)=k\}\ P\{x(t)=k|x(0)=i\}\\&=\sum_{k=0}^{\infty}P_{kj}(t)\ P_{ik}(t)\end{aligned}$$

and the proof is completed.

Hence, we have the following theorem.

Theorem 1. Differential equation for backward equation

For all states i, j and times $t\geq 0$

$P_{ij}(t)=\lambda_i P_{i+1j}(t)-\lambda_i P_{ij}(t)$

Proof: From Lemma 2, we obtain.

$$P_{ij}(t+h)-P_{ij}(t)=\sum_{k=0}^{\infty}P_{ik}(h)P_{kj}(t)-P_{ij}(t)$$

$$=\sum_{k\neq 0}^{\infty}P_{ik}(h)P_{kj}(t)-[1-P_{ii}(h)]P_{ij}(t)$$

ant thus

$$\lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=\lim_{h\to 0}\{\sum_{k=1}^{\infty}\frac{P_{ik}(h)}{h}P_{kj}(t)-[\frac{1-P_{ij}(h)}{h}]P_{ij}(t)\}$$

Now assuming that we can interchange the limit and the summation in the above and applying Lemma 1.

We obtain that

$$P_{ij}(t)=\nu_i\sum_{k=i}^{\infty}P_{ik}(h)P_{kj}(t)-\nu_iP_{ij}(t)$$

and the proof is completed

Hence, from Lemma 2 and theorem 1

We have the following the backward equations for the birth and death process.

$$P_{0j}'(t)=\nu_0P_{1t}(t)-\nu_0P_{0j}(t)$$

$$P_{ij}'(t)=(\nu_i+\mu_i)[\frac{\nu_i}{\nu_i+\mu_i}P_{i+1j}(t)+\frac{\mu_i}{\nu_i+\mu_i}P_{i-1j}(t)]-(\nu_i+\mu_i)P_{ij}(t)$$

Theorem 2. Differential equation for forward equation

For all states i,j and times $t\geq 0$.

$$P_{ij}'(t)=\sum_{k\neq j}\nu_kP_{kj}\ P_{ik}(t)-\nu_iP_{ij}(t)$$

Proof: From Lemma 2, we have

$$P_{ij}(t+h)-P_{ij}(t)=\sum_{k=0}^{\infty}P_{ik}(t)P_{kj}(h)-P_{ij}(t)$$

$$=\sum_{k\neq j}^{\infty}P_{ik}(t)P_{kj}(h)-[1-P_{jj}(h)]P_{ij}(t)$$

and thus

$$\lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=\lim_{h\to 0}\{\sum_{k\neq j}P_{ik}(t)\frac{P_{kj}(h)}{h}-[\frac{1-P_{ij}(h)}{h}]P_{ij}(t)$$

and assuming that we can interchange limit with summation, we obtain by Lemma 1 that

$$P_{ij}'(t)=\sum_{k\neq j}\nu_kP_{kj}\ P_{ik}(t)-\nu_iP_{ij}(t)$$

and the proof is completed.

hence, from Theorem 2, we have the following the forward equations for birth and death process.

The forward equations for thd birth and death process

$$P_{io}'(t)=\sum_{k\neq 0}(\lambda_k+\mu_k)P_{ko}P_{ik}(t)-\lambda_i P_{io}(t)$$

$$=(\nu_1+\mu_1)\frac{\mu_1}{\nu_1+\mu_1}P_{i1}(t)-\nu_o P_{io}(t)$$

$$=\mu_1 P_{i1}(t)-\nu_o P_{io}(t)$$

$$P_{ij}'(t)=\sum_{k\neq j}(\lambda_k+\mu_k)P_k P_{ik}(t)-(\lambda_i+\mu_i)P_{ij}(t)$$

$$=(\lambda_{j-1}+\mu_{j-1})\frac{\lambda_{j-1}}{\lambda_{j-1}\mu_{j-1}}P_{ij-1}(t)+(\lambda_{j+1}+\mu_{j+1})\frac{\mu_{j+1}}{\lambda_{j+1}+\mu_{j+1}}\times P_{ij+1}(t)$$

$$-(\lambda_j+\mu_j)P_{ij}(t)$$

$$=\lambda_{j-1}P_{ij-1}(t)+\mu_{j+1}P_{ij+1}(t)-(\lambda_j+\mu_j)\ P_{ij}(t)$$

3. Numerical Example

Consider a continious time Markov chain consisting of two states. Suppose a machine that works for an exponential amount of time having mean $1/\lambda$ before breaking down, and suppose that it makes an exponential amount of time having mean $1/\mu$ to repair the machine. If the machine is in working condition at time 0, then what is the probability that it will be working at time t=0?

To answer the above question, we note that the process is a birth and death process with state 0 meaning that the machine is working and state 1 that it is being repaired having parameters

$$\lambda_0=\lambda \qquad \mu_1=\mu$$

$$\lambda_i=0, \quad i\neq 0 \qquad \mu_i=0, \quad i\neq 1$$

From the backward equations for the birth and death process.

wed obtain,

(3.1) $P_{oo}'(t)=\lambda[P_{1o}(t)-P_{oo}(t)]$

(3.2) $P_{1o}'(t)=\mu P_{oo}(t)-\mu P_{1o}(t)$

multipling equation (3.1) by μ and equation (3.2) by λ and then adding the two equation yields,

$\mu P_{oo}(t)+\lambda P_{1o}(t)=0$

By integrating, we obtain that

$\mu P_{oo}'(t)+\lambda P_{1o}'(t)=C$ (C:constant)

However, since $P_{oo}(0)=1$ and $P_{1o}(0)=0$,

we obtain that $C=\mu$ and hence

(3.3) $\mu P_{oo}(t)+\lambda P_{1o}(t)=\mu$

or equivalently

$\lambda P_{1o}(t)=\mu[P_{oo}(t)]$

By substituting this result in equation (3.1), we obtain

$$P_{oo}'(t)=\mu[1-P_{oo}(t)]-\lambda P_{oo}(t)$$
$$=\mu-(\mu+\lambda)P_{oo}(t)$$

Let $h(t)=P_{oo}(t)-\dfrac{\lambda}{\mu+\lambda}$

$$h(t)=\mu-(\mu+\lambda)\left[h(t)+\frac{\lambda}{\mu+\lambda}\right]$$
$$=-(\mu+\lambda)\,h(t)$$
$$\frac{h'(t)}{h(t)}=-(\mu+\lambda)$$

By integrating both sides, we obtain

$$\log h(t)=-(\mu+\lambda)t+c$$

or $h(t)=K\cdot EXP(-(\mu+\lambda)t)$

and thus

$$P_{oo}(t)=K\cdot EXP(-(\mu+\lambda)t)-\frac{\mu}{\mu+\lambda}$$

Which finally yields, by setting $t=0$ and using the fact that $P_{oo}(0)=1$,that

$$P_{oo}(t)=\frac{\lambda}{\mu+\lambda}EXP(-(\mu+\lambda)t)+\frac{\mu}{\mu+\lambda}$$

From equation (3.3), this also implies that

$$P_{1o}(t)=\frac{\lambda}{\mu+\lambda}-\frac{\lambda}{\mu+\lambda}EXP(-(\mu+\lambda)t)$$

Hence, our desired probability $P_{oo}(10)$ equals

$$P_{oo}(10) = \frac{\lambda}{\mu+\lambda} EXP(-(\mu+\lambda)) + \frac{\mu}{\mu+\lambda}$$

(Examples)

1. $\mu=2$ $\lambda=3$

   $P_{oo}(10)=0.973$

2. $\mu=5$ $\lambda=4$

   $P_{oo}(10)=0.202$

3. $\mu=3$ $\lambda=5$

   $P_{oo}(10)=0.926$

4. Conclusion

As seen on the above numberical example, if the value of μ and λ is given, it is thought to be applied to real life, because a model of the most suitable value as to individual number process under continuous time Markov chain can be obtained.

Reference

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