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The Confidence Intervals for the Ratio of Total Variances in Three-Factor Nested Variance Component Model

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I. Introduction

Consider the three-factor nested variance component model given by

$$y_{ijkm} = \mu + A_i + B_{ij} + C_{ijk} + \varepsilon_{ijkm}, \tag{1.1}$$

where $i = 1, 2, \cdots, I$, $j = 1, 2, \cdots, J$, $k = 1, 2, \cdots, K$ and $m = 1, 2, \cdots, M$.

The A_i, B_{ij}, C_{ijk} and ε_{ijkm} are independent unobservable random variables and $A_i \sim N(0, \sigma_A^2)$, $B_{ij} \sim N(0, \sigma_B^2)$, $C_{ijk} \sim N(0, \sigma_C^2)$, $\varepsilon_{ijkm} \sim N(0, \sigma_\varepsilon^2)$, μ is an unknown parameter, and the y_{ijkm} are observable random variables. An analysis of variance for this model is displayed in Graybill in 1976.

The problem of confidence intervals on linear combinations of more than two variances is suggested by Smith in 1936. And in 1946, Satterthwaite studied and expanded the method and the result of his studies has been known as "Satterthwaite procedure".

Lately, the procedure has been widely used and developed by many authors. Especially, Howe (1974), Graybill and Wang (1979), Leiva and Graybill (1986) got a good approximation. The precisions of confidence interval are satisfactory, but it is the result mostly found out in one or two factor model.

In 1993, Kang is to use and expand of Gragbill and Wang's (1979) approximation, the approximation $1-\alpha$ lower and upper confidence intervals on σ_A^2, σ_B^2, σ_C^2, σ_ε^2 and the ratio of these of (1.1) in three-factor.

The purpose of this paper, there is no method available for setting exact $1-\alpha$ confidence intervals on the ratio of total variances of these, so by using this approximate $1-\alpha$ confidence intervals and the precisions will be treated in Chapter III and IV.

II. Three-factor Nested Variance Component Model

The model is (1.1). And $E[A_i] = E[B_{ij}] = E[C_{ijk}] = E[\varepsilon_{ijkm}] = 0, \mathrm{Var}[A_i] = \sigma_A^2, \mathrm{Var}[B_{ij}] = \sigma_B^2, \mathrm{Var}[C_{ijk}] = \sigma_C^2, \mathrm{Var}[\varepsilon_{ijkm}] = \sigma_\varepsilon^2$. The ANOVA table of (1.1) is as follows :

< Table 1 >

Source	D.F.	S.S.	M.S.	E.M.S.
Total	$IJKM$	$\sum\sum\sum\sum y_{ijkm}^2$		
Mean	1	$y^2/(IJKM)$		
Factor A	n_1	$\sum\sum\sum\sum(\bar{y}_{i...} - \bar{y}_{....})^2$	S_1^2	θ_1
B within A	n_2	$\sum\sum\sum\sum(\bar{y}_{ij..} - \bar{y}_{i...})^2$	S_2^2	θ_2
C within B within A	n_3	$\sum\sum\sum\sum(\bar{y}_{ijk.} - \bar{y}_{ij..})^2$	S_3^2	θ_3
Error	n_4	$\sum\sum\sum\sum(y_{ijkm} - \bar{y}_{ijk.})^2$	S_4^2	θ_4

where

$$n_1 = I-1, \quad n_2 = I(J-1), \quad n_3 = IJ(K-1), \quad n_4 = IJK(M-1),$$

$$\theta_1 = \sigma_\varepsilon^2 + M\sigma_C^2 + KM\sigma_B^2 + JKM\sigma_A^2, \quad \theta_2 = \sigma_\varepsilon^2 + M\sigma_C^2 + KM\sigma_B^2,$$
$$\theta_3 = \sigma_\varepsilon^2 + M\sigma_C^2, \quad \theta_4 = \sigma_\varepsilon^2.$$

The upper α probability point of Snedecor's F distribution and chi-square distribution are denoted by $F_{\alpha;\cdot,\cdot}$ and $\chi^2_{\alpha;\cdot}$, respectively, where $\alpha_1 + \alpha_2 = \alpha$, often $\alpha_1 = \alpha_2 = \alpha/2$.

From the definition of chi-square distribution, we get

$$\frac{n_4 S_4^2}{\sigma_\varepsilon^2} = \chi^2_{n_4}, \quad \mathrm{P}\left[\frac{S_4^2}{\theta_4} \le \frac{\chi^2_{\alpha;n_4}}{n_4} = F_{\alpha;n_4,\infty}\right] = 1-\alpha.$$

III. The Confidence Intervals on the Ratio of Variances

3.1. Confidence intervals on $\sigma_A^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$

At first, we want to find out the form of the $1-\alpha$ upper confidence interval on $\frac{\sigma_A^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2}$.

From Table 1, we get

$$\frac{JKM\sigma_A^2}{JKM(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)}$$
$$= \frac{\theta_1-\theta_2}{\theta_1+(J-1)\theta_2+J(K-1)\theta_3+JK(M-1)\theta_4}.$$

In this, we consider, the lower limit of $1-\alpha$ upper confidence interval on $\frac{J\sigma_A^2}{\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2}$. Then it is a function (Graybill and Wang) of S_1^2, S_2^2, S_3^2 and S_4^2 say $q(S_1^2, S_2^2, S_3^2, S_4^2)$, such that

$$\text{(3.1)} \qquad \mathrm{P}\left[q \le \frac{\theta_1-\theta_2}{\theta_2+(K-1)\theta_3+K(M-1)\theta_4}\right] = 1-\alpha.$$

We want the confidence interval to be an exact $1-\alpha$ interval.

1) When $\sigma_C^2 = \sigma_\varepsilon^2 = 0$ (i.e. $\theta_3 = 0$, $\theta_4 = 0$), we get

$$\frac{\theta_1-\theta_2}{\theta_2+(K-1)\theta_3+K(M-1)\theta_4} = \frac{\theta_1}{\theta_2}-1$$

and

$$\mathrm{P}\left[\frac{S_1^2}{S_2^2 F_{\alpha;n_1,n_2}}-1 \le \frac{\theta_1}{\theta_2}-1\right] = 1-\alpha.$$

2) When the hypothesis $\mathrm{H}_0 : \sigma_A^2 = 0$ is accepted for a size α test, we want the corresponding $1-\alpha$ confidence interval to include zero, and when H_0 is rejected we want q to be an increasing function of S_1^2/S_2^2. This condition is required because $\frac{\theta_1-\theta_2}{\theta_2+(K-1)\theta_3+K(M-1)\theta_4}$ in (3.1) is an increasing function of θ_1/θ_2, and if θ_1/θ_2 increases then its estimate would tend to increase. Hence if q does not increase as S_1^2/S_2^2 increases, clearly the confidence coefficient would deviate a large mount from $1-\alpha$ as θ_1/θ_2 gets large.

For a size α test of $\mathrm{H}_0 : \sigma_A^2 = 0$ versus $\mathrm{H}_a : \sigma_A^2 > 0$, the hypothesis H_0 is accepted if and only if $\frac{S_1^2}{S_2^2} \le F_{\alpha;n_1,n_2}$. Thus, we want

$$\begin{cases} q(S_1^2, S_2^2, S_3^2, S_4^2) = 0 & \text{when } S_1^2 \le S_2^2 F_{\alpha;n_1,n_2} \\ q(S_1^2, S_2^2, S_3^2, S_4^2) > 0 & \\ \quad \text{and increasing in } S_1^2/S_2^2 & \text{when } S_1^2 > S_2^2 F_{\alpha;n_1,n_2} \end{cases}$$

3) When $J \to \infty$ (hence n_2, n_3, and $n_4 \to \infty$), we want the confidence interval to have an exact confidence coefficient $1-\alpha$. When $J \to \infty$, it follows that $n_2 \to \infty$, $n_3 \to \infty$ and $n_4 \to \infty$ and from this it follows that $S_2^2 \to \theta_2$, $S_3^2 \to \theta_3$ and $S_4^2 \to \theta_4$ in probability also,

$$S_2^2 + (K-1)S_3^2 + K(M-1)S_4^2 \to \theta_2 + (K-1)\theta_3 + K(M-1)\theta_4.$$

From these results, we get

$$\mathrm{P}\left[\frac{S_1^2}{F_{\alpha;n_1,\infty}} \le \theta_1\right] = \mathrm{P}\left[\frac{S_1^2}{F_{\alpha;n_1,\infty}} - S_2^2 \le \theta_1 - \theta_2\right]$$
$$= 1-\alpha.$$

Then divide the left and right sides, respectively, by

$$S_2^2 + (K-1)S_3^2 + K(M-1)S_4^2$$

and its equivalent value $\theta_2 + (K-1)\theta_3 + K(M-1)\theta_4$ to obtain

$$\mathrm{P}\left[\frac{S_1^2 - S_2^2 F_{\alpha;n_1,\infty}}{\{S_2^2 + (K-1)S_3^2 + K(M-1)S_4^2\}F_{\alpha;n_1,\infty}} \le \frac{\theta_1 - \theta_2}{\theta_2 + (K-1)\theta_3 + K(M-1)\theta_4}\right]$$
$$= 1-\alpha.$$

Hence, when $J \to \infty$, we obtain

$$q = \begin{cases} \dfrac{S_1^2 - S_2^2 F_{\alpha;n_1,\infty}}{\{S_2^2 + (K-1)S_3^2 + K(M-1)S_4^2\}F_{\alpha;n_1,\infty}} & \text{when } \dfrac{S_1^2}{S_2^2} \ge F_{\alpha;n_1,\infty} \\ 0 & \text{when } \dfrac{S_1^2}{S_2^2} < F_{\alpha;n_1,\infty} \end{cases}$$

The maximum likelihood estimate of

$$\frac{\theta_1 - \theta_2}{\theta_2 + (K-1)\theta_3 + K(M-1)\theta_4} \quad \left(= \frac{J\sigma_A^2}{\sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2}\right)$$

is the form $\frac{a_0 S_1^2 + a_1 S_2^2}{a_2 S_2^2 + a_3 S_3^2 + a_4 S_4^2}$, and determine the constants a_0, a_1, a_2, a_3 and a_4 by imposing the conditions 1), 2) and 3), we get $a_0 = 1$, $a_2 = -a_1 = -F_{\alpha;n_1,n_2}$. Thus,

$$q = \begin{cases} \dfrac{S_1^2 - S_2^2 F_{\alpha;n_1,n_2}}{S_2^2 F_{\alpha;n_1,n_2} + a_3^* S_3^2 + a_4^* S_4^2} & \text{when } \dfrac{S_1^2}{S_2^2} > F_{\alpha;n_1,n_2} \\ 0 & \text{when } \dfrac{S_1^2}{S_2^2} \le F_{\alpha;n_1,n_2}, \end{cases}$$

where

$$\lim_{J\to\infty} a_3^* = (K-1)F_{\alpha;n_1,\infty}, \quad \lim_{J\to\infty} a_4^* = K(M-1)F_{\alpha;n_1,\infty}.$$

Hence, we choose

$$a_3^* = (K-1)F_{\alpha;n_1,n_3}, \quad a_4^* = K(M-1)F_{\alpha;n_1,n_4}.$$

Let

$$A_\alpha = (J-1)S_2^2 F_{\alpha;n_1,n_2}, \quad B_\alpha = J(K-1)S_3^2 F_{\alpha;n_1,n_3}$$

and

$$C_\alpha = JK(M-1)F_{\alpha;n_1,n_4}.$$

The lower limit of a $1-\alpha$ upper confidence interval on

$$\frac{\theta_1 - \theta_2}{\theta_1 + (J-1)\theta_2 + J(K-1)\theta_3 + JK(M-1)\theta_4} = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2}$$

is

$$
(3.2) \qquad L = \begin{cases} \dfrac{S_1^2 - S_2^2 F_{\alpha;n_1,n_2}}{S_1^2 + A_\alpha + B_\alpha + C_\alpha} & \text{when } \dfrac{S_1^2}{S_2^2} > F_{\alpha;n_1,n_2} \\ 0 & \text{when } \dfrac{S_1^2}{S_2^2} \le F_{\alpha;n_1,n_2}, \end{cases}
$$

And the upper limit of a $1-\alpha$ lower confidence interval is obtained from (3.2) by replacing α with $1-\alpha$ in the tabulated F's. We get

$$
\mathrm{P}\left[\frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2} \le U\right] = 1 - \alpha,
$$

where

$$
U = \begin{cases} \dfrac{S_1^2 - S_2^2 F_{1-\alpha;n_1,n_2}}{S_1^2 + A_{1-\alpha} + B_{1-\alpha} + C_{1-\alpha}} & \text{when } \dfrac{S_1^2}{S_2^2} > F_{1-\alpha;n_1,n_2} \\ 0 & \text{when } \dfrac{S_1^2}{S_2^2} \le F_{1-\alpha;n_1,n_2}. \end{cases}
$$

3.2. Confidence intervals on $\sigma_B^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2)$

Similar to that of Section 3.1, we can find out the upper confidence interval on

$$
\frac{KM\sigma_B^2}{JKM(\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2)} = \frac{\theta_2 - \theta_3}{\theta_1 + (J-1)\theta_2 + J(K-1)\theta_3 + JK(M-1)\theta_4}.
$$

Let

$$D_\alpha = S_2^2 - S_3^2 F_{\alpha;n_2,\infty} + F_{\alpha;n_2,n_3} + (F_{\alpha;n_2,\infty} - F_{\alpha;n_2,n_3})\frac{S_3^4}{S_2^2},$$

$$E_\alpha = \frac{S_1^2}{F_{1-\alpha;n_1,\infty}} + (J-1)S_2^2 + J(K-1)S_3^2 F_{\alpha;n_2,\infty} + JK(M-1)S_4^2.$$

The lower limit of a $1-\alpha$ upper confidence interval on

$$\frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2}$$

is

$$L = \begin{cases} \min\left[1, \dfrac{JD_\alpha}{E_\alpha}\right] & \text{if } \dfrac{S_2^2}{S_3^2} > F_{\alpha;n_2,n_3} \\ 0 & \text{if } \dfrac{S_2^2}{S_3^2} \le F_{\alpha;n_2,n_3}. \end{cases}$$

And by using a procedure similar to the one used to determine an approximate a $1-\alpha$ lower confidence interval on $\frac{\sigma_B^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2}$ is

$$\mathrm{P}\left[\frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2} \le U\right] = 1-\alpha,$$

where

$$U = \begin{cases} \min\left[1, \dfrac{JD_{1-\alpha}}{E_{1-\alpha}}\right] & \text{if } \dfrac{S_2^2}{S_3^2} > F_{1-\alpha;n_2,n_3} \\ 0 & \text{if } \dfrac{S_2^2}{S_3^2} \le F_{1-\alpha;n_2,n_3}. \end{cases}$$

3.3. Confidence intervals on $\sigma_C^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$

The lower limit of a $1-\alpha$ upper confidence intervals on

$$\frac{\sigma_C^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2}.$$

We can shows the similar to those of Section 3.1.

Putting

$$T_{\alpha,1}=\frac{S_1^2}{F_{1-\alpha;n_1,n_3}},\quad T_{\alpha,2}=\frac{(J-1)S_2^2}{F_{1-\alpha;n_2,n_3}}$$

and

$$T_{\alpha,3}=J(K-1)S_3^2+JK(M-1)S_4^2F_{\alpha;n_3,n_4},$$

we get

$$q=\frac{S_3^2-S_4^2F_{1-\alpha;n_3,n_4}}{T_{\alpha,1}+T_{\alpha,2}+T_{\alpha,3}}.$$

Therefore, the lower limit of $1-\alpha$ upper confidence interval on

$$\frac{\sigma_C^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2}$$

is

$$L=\begin{cases}\min\left[1,\dfrac{JK(S_3^2-S_4^2F_{\alpha;n_3,n_4})}{T_{\alpha,1}+T_{\alpha,2}+T_{\alpha,3}}\right] & \text{if } \dfrac{S_3^2}{S_4^2}>F_{\alpha;n_3,n_4}\\ 0 & \text{if } \dfrac{S_3^2}{S_4^2}\le F_{\alpha;n_3,n_4}.\end{cases}$$

The upper limit of a $1-\alpha$ lower confidence interval is obtained from L by replacing α with $1-\alpha$ in the tabulated F's. Thus we get

$$\mathrm{P}\left[\frac{\theta_3-\theta_4}{\theta_1+(J-1)\theta_2+J(K-1)\theta_3+JK(M-1)\theta_4}\le U\right],$$

where

$$U = \begin{cases} \min\left[1, \dfrac{JK(S_3^2 - S_4^2 F_{1-\alpha;n_3,n_4})}{T_{1-\alpha,1} + T_{1-\alpha,2} + T_{1-\alpha,3}}\right] & \text{if } \dfrac{S_3^2}{S_4^2} > F_{1-\alpha;n_3,n_4} \\ 0 & \text{if } \dfrac{S_3^2}{S_4^2} \le F_{1-\alpha;n_3,n_4}. \end{cases}$$

3.4. Confidence intervals on $\sigma_\varepsilon^2/(\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2)$

We can begin with

$$\frac{JKM(\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + \sigma_\varepsilon^2)}{\sigma_\varepsilon^2} = \frac{\theta_1 + (J-1)\theta_2 + J(K-1)\theta_3 + JK(M-1)\theta_4}{\theta_4},$$

we want to determine a function of S_1^2, S_2^2 and S_4^2 say $q(S_1^2, S_2^2, S_4^2)$ such that $\mathrm{P}\left[q \le \frac{\theta_1+(J-1)\theta_2}{\theta_4}\right]$ is approximately equal to $1-\alpha$. Scale invariance and large sample theory suggested that $q = \frac{S_2^2}{S_4^2} h\left(\frac{S_1^2}{S_2^2}\right)$, where the function $h(\cdot)$ is determined by requiring it to satisfy the following three conditions :

1) When the hypothesis $\mathrm{H}_0 : \theta_1 = \theta_2$ is accepted by using a size α test (i.e. $S_1^2/S_2^2 \le F_{\alpha;n_1,n_2}$) we require the corresponding $1-\alpha$ confidence interval on $\frac{\theta_1+(J-1)\theta_2}{\theta_4}$ to be exact. The exact $1-\alpha$ confidence interval on $(J-1)\theta_2/\theta_4$ is

$$\frac{S_2^2(J-1)\left(n_1\frac{S_1^2}{S_2^2} + n_2\right)}{S_4^2(n_1+n_2)F_{\alpha;n_1+n_2,n_4}} \le \frac{(J-1)\theta_2}{\theta_4}.$$

2) The confidence coefficient for the interval on $\frac{\theta_1+(J-1)\theta_2}{(J-1)\theta_4}$ is to be exact as $J \to \infty$. To satisfy this we let

$$h\left(\frac{S_1^2}{S_2^2}\right) = \frac{S_1^2}{S_2^2} F_{\alpha;n_1,\infty} + J - 2 + g\left(\frac{S_1^2}{S_2^2}\right),$$

where

$$\lim_{J\to\infty} \frac{1}{Jg\left(\frac{S_1^2}{S_2^2}\right)} = 0.$$

3) The confidence coefficient is to be exact as $\sigma_A^2 \to \infty$; when $\sigma_A^2 \to \infty$ the quantity $\frac{\sigma_\epsilon^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\epsilon^2}$ is dominated by θ_1/θ_4 so we want an exact $1-\alpha$ confidence interval on θ_1/θ_4. Thus we get $h\left(\frac{S_1^2}{S_2^2}\right) = \frac{S_1^2}{S_2^2} F_{\alpha;n_1,n_4}$ when $\sigma_A^2 \to \infty$ (and hence $S_1^2 \to \infty$).

If we impose conditions 1), 2) and 3), we get the following for the approximate upper $1-\alpha$ confidence interval on $\frac{\theta_1+(J-1)\theta_2}{\theta_4}$ is

$$\frac{\theta_1 + (J-1)\theta_2}{\theta_4} \leq \begin{cases} \dfrac{S_2^2}{S_4^2 F_{1-\alpha;n_1,n_4}} \left[\dfrac{S_1^2}{S_2^2} + X_{1-\alpha}\right] & \text{if } \dfrac{S_1^2}{S_2^2} > F_{1-\alpha;n_1,n_2} \\ \dfrac{S_2^2}{S_4^2} \dfrac{J\left(n_1 \frac{S_1^2}{S_2^2} + n_2\right)}{(n_1+n_2)F_{1-\alpha;n_1+n_2,n_4}} & \text{if } \dfrac{S_1^2}{S_2^2} \leq F_{1-\alpha;n_1,n_2}, \end{cases} \tag{3.3}$$

where

$$X_{1-\alpha} = \frac{J(n_1 F_{1-\alpha;n_1,n_2} + n_2)F_{1-\alpha;n_1,n_4}}{(n_1+n_2)F_{1-\alpha;n_1+n_2,n_4}} - F_{1-\alpha;n_1,n_2}.$$

By a simulation study, we determined that the confidence coefficients for the interval in (3.3) are too large, so the following function was considered :

$$h\left(\frac{S_1^2}{S_2^2}\right) = b_0 + b_1\left(\frac{S_1^2}{S_2^2}\right) + b_2\left(\frac{S_1^2}{S_2^2}\right)^{-1},$$

which would tend to decrease the confidence coefficients in (3.3). We now impose conditions 1), 2) and 3) to evaluate b_0, b_1 and b_2. So we get

$$\frac{\theta_1 + (J-1)\theta_2}{\theta_4} \le \begin{cases} \dfrac{S_2^2}{S_4^2 F_{1-\alpha;n_1,n_4}}\left[\dfrac{S_1^2}{S_2^2} + Y + F_{1-\alpha;n_1,n_2}(X_{1-\alpha} - Y)\dfrac{S_2^2}{S_1^2}\right] & \text{if } \dfrac{S_1^2}{S_2^2} > F_{1-\alpha;n_1,n_2} \\ V_{1-\alpha}\dfrac{1}{S_4^2} & \text{if } \dfrac{S_1^2}{S_2^2} \le F_{1-\alpha;n_1,n_2}, \end{cases}$$

where

$$Y = \frac{J F^2_{1-\alpha;n_1,\infty}}{F_{1-\alpha;n_1,n_2}} - \frac{F_{1-\alpha;n_1,n_4}}{F_{1-\alpha;n_1+n_2,\infty}},$$

$$V_{1-\alpha} = \frac{S_2^2\left(n_1\frac{S_1^2}{S_2^2} + n_2\right)}{(n_1+n_2)F_{1-\alpha;n_1+n_2,n_4}}.$$

Thus, the $1-\alpha$ upper confidence interval on $\frac{\sigma_\epsilon^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\epsilon^2}$ is

$$L = \begin{cases} \min\{1, S_{1-\alpha}\} & \text{if } S_1^2/S_2^2 \ge F_{1-\alpha;n_1,n_2} \\ \min\{1, T_{1-\alpha}\} & \text{if } S_1^2/S_2^2 < F_{1-\alpha;n_1,n_2}, \end{cases}$$

where

$$S_{1-\alpha} = \frac{JKS_3^2}{\frac{S_2^2}{F_{1-\alpha;n_1,n_3}}\left[\frac{S_1^2}{S_2^2} + Y + F_{1-\alpha;n_1,n_2}(X_{1-\alpha} - Y)\frac{S_2^2}{S_1^2}\right] + J(K-1)S_3^2},$$

$$T_{1-\alpha} = \frac{KS_3^2}{V_{1-\alpha} + (K-1)S_3^2}.$$

By using a procedure similar to the one used to determine an approximate $1-\alpha$ upper confidence interval on $\frac{\sigma_\varepsilon^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2}$ is

$$\mathrm{P}\left[\frac{\sigma_\varepsilon^2}{\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2} \le U\right] = 1-\alpha,$$

where

$$U = \begin{cases} \min\{1, S_\alpha\} & \text{if } s_1^2/S_2^2 \ge F_{\alpha;n_1,n_2} \\ \min\{1, T_\alpha\} & \text{if } s_1^2/S_2^2 < F_{\alpha;n_1,n_2}, \end{cases}$$

$$S_\alpha = \frac{JKS_3^2}{\frac{S_2^2}{F_{\alpha;n_1,n_3}}\left[\frac{S_1^2}{S_2^2} + \frac{1}{2}(W+X) + \frac{1}{2}F_{\alpha;n_1,n_2}(X-W)\frac{S_2^2}{S_1^2}\right] + J(K-1)S_3^2},$$

$$T_\alpha = \frac{KS_3^2}{V_\alpha + (K-1)S_3^2},$$

$$W = (J-1)F_{\alpha;n_1,\infty},$$

$$X = \frac{J(n_1F_{\alpha;n_1,n_2} + n_2)F_{\alpha;n_1,n_4}}{(n_1+n_2)F_{\alpha;n_1+n_2,n_4}} - F_{\alpha;n_1,n_2},$$

$$V_\alpha = \frac{S_2^2\left(n_1\frac{S_1^2}{S_2^2} + n_2\right)}{(n_1+n_2)F_{\alpha;n_1+n_2,n_4}}.$$

IV. Discussion

In this chapter, I would like to discuss these precisions.

(1) The method of Graybill and Wang approximation was used in finding out the confidence interval on $\sigma_A^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ of (1.1). Graybill and Wang confidence intervals of the two-factor nested variance model, when $1-\alpha=0.95$, the precisions of the confidence intervals L and U lie between 0.950-0.966, 0.950-0.953 respectively.

Accordingly the precisions of the above-mentioned intervals will be similar to those of this.

(2) The confidence intervals on $\sigma_B^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ and $\sigma_C^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ are similar to that of Graybill and Wang's confidence intervals on $\sigma_B^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2)$. Thus, the precisions of the confidence intervals on $\sigma_B^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ and $\sigma_C^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ similar to that of this, when $1-\alpha=0.95$, the precisions of the confidence interval L and U lie between 0.947-0.986, 0.950-0.968 respectively.

(3) The precisions of the confidence intervals on $\sigma_\varepsilon^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ is similar to that of Graybill and Wang's precisions of the confidence intervals on $\sigma_C^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2)$. Thus the precisions of the confidence intervals on $\sigma_\varepsilon^2/(\sigma_A^2+\sigma_B^2+\sigma_C^2+\sigma_\varepsilon^2)$ is similar to that of od this, when $1-\alpha=0.95$, the precisions of the confidence interval L and U lie between 0.947-0.979, 0.946-0.957 respectively.

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