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A Study on Multiobjective Fractional Optimization Problems

Chang Wook Kim

Department of Applied Mathematics
Korea Maritime University, Pusan, Korea

1. Introduction

Duality theorems for single objective fractional optimization problems have been of much interest in the past [1, 4, 8, 10, 11, 12]. Recently there has been of growing interest in studying optimality theorems and duality theorems for multiobjective fractional optimization problem [2, 3, 5, 13, 14, 15]. In particular, Singh [12] considered a nondifferentiable single objective fractional optimizational problem in which numerators of objective functions involves the square roots of quadratic forms and established optimality theorems and duality theorems in the framework of the Hanson-Mond classes of functions. Also Bhatia and Jain [1] extended Singh's results to a nondifferentiable multiobjective fractional optimization problem in which numerators of objective functions involves the square roots of quadratic forms.

In this paper, we consider the following nondifferentiable multiobjective optimization problem (P)

$$\text{(P)} \qquad \text{Minimize } F(x) = \left(\frac{f_1(x) + (xD_1x)^{1/2}}{h_1(x)} \cdots \frac{f_k(x) + (xD_kx)^{1/2}}{h_k(x)} \right)$$

$$\text{subject to } g(x) \leq 0, \qquad x \in X,$$

where X is an open convex subset of R^n, each $f_i : X \to R$, $h_i : X \to R$, $i = 1, 2, \cdots, k$, $g : X \to R^m$ are differentiable and D_i, $i = 1, 2, \cdots, k$ are symmetric positive semidefinite matrices. Let X_0 denote the set of feasible solutions of problem (P). We assume that X_0 is compact and $h_i(x) > 0$ on X_0, $i = 1, 2, \cdots k$. All vectors are considered to be column vectors. For simplicity, we avoid the use of the superscript t over a vector to label it as row vector. For instance, instead of writing $x^t D_i x$, we simply write xD_ix, etc.

A sufficient optimality theorem for a feasible solution of (P) to be property efficient is given. A dual problem for (P) is considered and certain duality theorems are obtained under the ρ-convexity assumptions.

2. Preliminaries

Now we give the definitions and results needed in later sections.

Definition 2.1. A feasible solution $x^0 \in X_0$ is said to be an efficient solution of (P) if there exists no other feasible solution $x \in X_0$ such that

$$F_i(x) \leq F_i(x^0), \quad i = 1, 2, \cdots, k, \quad F(x) \neq F(x^0).$$

Definition 2.2. A feasible solution $x^0 \in X_0$ is said to be an properly efficient of (P) if it is efficient for (P) and if there exists a scalar $M > 0$ such that for each i,

$$\frac{f_i(x^0) - f_i(x)}{f_j(x) - f_j(x^0)} \leq M$$

for some j, $F_j(x) > F_j(x^0)$ whenever x is feasible for (P) and $F_i(x) < F_i(x^0)$.

Lemma 2.1 [6]. Let D be an $n \times n$ real, symmetric, positive semidefinite matrix. Then, for any $x \in R^n$, $y \in R^n$,

$$xDy \leq (xDx)^{1/2}(yDy)^{1/2}.$$

Lemma 2.2 [2]. Let $x^0 \in X_0$. If (x^0, y^0) is properly efficient for the following multiobjective optimization problem (P^1) with $y = y^0$, where

$$y_i^0 = \frac{f_i(x^0) + (x^0 D_i x^0)^{1/2}}{h_i(x^0)}, \qquad i = 1, 2, \cdots, k,$$

then x^0 is properly efficient for (P).

$$(\mathrm{P}^1) \qquad \text{Minimize } [f_1(x) + (xD_1x)^{1/2} - y_1 h_1(x), \cdots, \\ f_k(x) + (xD_kx)^{1/2} - y_k h_k(x)]$$

$$\text{subject to } g(x) \leq 0, \qquad x \in X, \quad y \in R^k.$$

Theorem 2.1 [2]. Suppose that x^0 is properly efficient solution of (P) and the set Z^0 is empty. Then there exists $\lambda_i > 0$, $i = 1, 2, \cdots, k$,

$$\sum_{i=1}^{k} \lambda_i = 1,\ y_i^0 \geq 0,\ i = 1, 2, \cdots k,\ v^0 \in R^m, v_j^0 \geq 0,\ w_i^0 \in R^n,\ i = 1, 2, \cdots, k,$$

such that

$$\sum_{i=1}^{k} \lambda[\nabla f_i(x^0) + D_i w_i^0 - \nabla y_i^0 h_i(x^0)] + \nabla v^0 g(x^0) = 0$$

$$v^0 g(x^0) \geq 0,$$

$$w_i^0 D_i w_i^0 \leq 1,\ i = 1, 2, \cdots, k$$

$$(x^0 D_i x^0)^{1/2} = x^0 D_i w_i^0,\ i = 1, 2, \cdots, k.$$

For a feasible solution x^0 of (P), following Mond and Schechter [9], we define

$$Z^0 = \cup_{i=1}^{k} Z_i^0,$$

where

$$Z_i^0 = \{z : z\nabla g_j(x^0) \le 0 \text{ for all } j \in Q \text{ and}$$
$$z[\nabla f_i(x^0) - \nabla y_i^0 h_i(x^0)] + zD_i x^0 / x^0 D_i x^0 < 0 \quad \text{if } x^0 D_i x^0 > 0,$$
$$z[\nabla f_i(x^0) - \nabla y_i^0 h_i(x^0)] + (zD_i z < 0 \text{ if } x^0 D_i x^0 = 0\}$$

for $i = 1, 2, \cdots, k$, where $Q = \{j : g_j(x^0) = 0\}$ and

$$y_i^0 = \frac{f_i(x^0) + (x^0 D_i x^0)^{1/2}}{h_i(x^0)} \qquad i = 1, 2, \cdots, k.$$

Definition 2.3. Let f be a real valued differentiable function defined on a subset X of R^n. Then f is said to be ρ-convex if there exists some real number ρ such that for each x, $u \in X$,

$$f(x) - f(u) \ge (x - u)\nabla f(u) + \rho \|x - u\|^2.$$

3. Sufficient optimality theorem

Now we establish a sufficient optimality theorem for (P) under the ρ-convexity assumptions

Theorem 3.1. Suppose that there exists a feasible solution x^0 of (P) and a scalar $\lambda_i > 0$, $i = 1, 2, \cdots, k$, $\sum_{i=1}^{k} \lambda_i = 1$, $y_i^0 \ge 0$, $i = 1, 2, \cdots, k$, $v^0 \in R^m$, $v^0 \ge 0$, $w_i^0 \in R^n$, $i = 1, 2, \cdots, k$ with

$$y_i^0 = \frac{f_i(x^0) + (x^0 D_i x^0)^{1/2}}{h_i(x^0)}, \ i = 1, 2, \cdots, k$$

such that

$$\sum_{i=1}^{k} \lambda_i[\nabla f_i(x^0) + D_i w_i^0 - \nabla y_i^0 h_i(x^0)] + \nabla v^0 g(x^0) = 0 \tag{1}$$

$$v^0 g(x^0) = 0, \tag{2}$$

$$w_i^0 D_i w_i^0 \leq 1, \ i = 1, 2, \cdots, k, \tag{3}$$

$$(x^0 D_i x^0)^{1/2} = x^0 D_i w_i^0, \ i = 1, 2, \cdots, k. \tag{4}$$

Further suppose that f_i is ρ_i-convex, $-h_i$ is ρ_i^*-convex, $i = 1, 2, \cdots, k$, and g_i is ρ_j^{**}-convex, $j = 1, 2, \cdots, m$ and that

$$\sum_{i=1}^{k}(\lambda_i \rho_i - \lambda_i y_i^0 \rho_i^*) + \sum_{j=1}^{m} v_j^0 \rho_j^{**} \geq 0, \tag{5}$$

Then x^0 is a properly efficient solution of (P).

Proof. By (1), (5) and the ρ-convexity assumptions, we have

$$0 \leq \sum_{i=1}^{k} \lambda_i[f_i(x) - f_i(x^0)] + \sum_{i=1}^{k} \lambda_i(x - x^0) D_i w_i^0$$
$$- \sum_{i=1}^{k} \lambda_i y_i^0 [h_i(x) - h_i(x^0)] + \sum_{j=1}^{m} v_j^0 [g_j(x) - g_j(x^0)].$$

By (2), $v_j^0 g_j(x^0) = 0$ and since x is feasible, $v_i^0 g_j(x) \leq 0$, $j = 1, 2, \cdots, m$. Hence (6) reduces to

$$0 \leq \sum_{i=1}^{k} \lambda_i[f_i(x) - f_i(x^0)] + \sum_{i=1}^{k} \lambda_i(x - x^0) D_i w_i^0$$
$$- \sum_{i=1}^{k} \lambda_i y_i^0 [h_i(x) - h_i(x^0)].$$

By (3), (4) and Lemma 2.1, we have

$$0 \leq \sum_{i=1}^{k} \lambda_i [f_i(x) - f_i(x^0)] + \sum_{i=1}^{k} \lambda_i [(xD_i x)^{1/2} - (x^0 D_i x^0)^{1/2}]$$

$$- \sum_{i=1}^{k} \lambda_i y_i^0 [h_i(x) - h_i(x^0)].$$

Hence we have

$$\sum_{i=1}^{k} \lambda_i [f_i(x^0) + (x^0 D_i x^0)^{1/2} - y_i^0 h_i(x^0)]$$

$$\leq \sum_{i=1}^{k} \lambda_i [f_i(x) + (xD_i x)^{1/2} - y_i^0 h_i(x)].$$

By Theorem 1 in [7], (x^0, y^0) is properly efficient for (P^1). By Lemma 2.2, x^0 is a properly efficient solution of (P).

4. Duality Theorems

Now we give the dual problem (D) for (P).

(D) Maximize $G(s, v, y, w_1, \cdots, w_k) = y = (y_1, \cdots, y_k)$

subject to

$$\sum_{i=1}^{k} \lambda_i [\nabla f_i(s) + D_i w_i - \nabla y_i h_i(s)] + \nabla v^t g(s) = 0 \tag{7}$$

$$f_i(s) + (sD_i s)^{1/2} - y_i h_i(s) \geq 0, \; i = 1, 2, \cdots, k, \tag{8}$$

$$w_i D_i w_i \leq 1, \; i = 1, 2, \cdots, k, \tag{9}$$

$$(sD_i s)^{1/2} = sD_i w_i, \; i = 1, 2, \cdots, k. \tag{10}$$

$$vg(s) \geq 0, \tag{11}$$

$$v \geq 0, \; y \geq 0, \tag{12}$$

where $\lambda_i > 0$, $i = 1, 2, \cdots k$ and $\sum_{i=1}^{k} \lambda_i = 1$.

Theorem 4.1. Let x be any feasible solution of (P) and $(s, v, y, w_1, \cdots, w_k)$ be any feasible solution of (D) for any $\lambda > 0$. Suppose that f_i is ρ_i-convex, $-h_i$ is ρ_i^*-convex, $i = 1, 2, \cdots, k$, and g_j is ρ_j^{**}-convex, $j = 1, 2, \cdots, m$ and that

$$\sum_{i=1}^{k} (\lambda_i \rho_i - \lambda_i y_i \rho_i^*) + \sum_{j=1}^{m} v_j \rho_j^{**} \geq 0.$$

Then the following does not holds:

$$\frac{f_i(x) + (xD_i x)^{1/2}}{h_i(x)} \leq y_i \text{ for all } i = 1, 2, \cdots, k$$

and

$$\frac{f_j(x) + (xD_j x)^{1/2}}{h_j(x)} < y_j \text{ for some } j.$$

Proof. Suppose that the following holds;

$$\frac{f_i(x) + (xD_i x)^{1/2}}{h_i(x)} \leq y_i \text{ for all } i = 1, 2, \cdots, k$$

and

$$\frac{f_j(x) + (xD_j x)^{1/2}}{h_j(x)} < y_j \text{ for some } j.$$

Then by (8), (11) and (12), we have

$$
\begin{aligned}
0 > & \sum_{i=1}^{k} \lambda_i [f_i(x) + (xD_i x)^{1/2} - y_i h_i(x)] \\
& - \sum_{i=1}^{k} \lambda_i [f_i(s) + (sD_i s)^{1/2} - y_i h_i(s)] + vg(x) - vg(s) \\
\geq & \sum_{i=1}^{k} \lambda_i [(x-s)\nabla f_i(s) + \rho_i \|x-s\|^2] \\
& - \sum_{i=1}^{k} \lambda_i [(x-s)\nabla y_i h_i(s) + \rho_i^* \|x-s\|^2] \\
& + \sum_{j=1}^{m} [(x-s)\nabla v_j g_j(s) + v_j^{**} \rho_j \|x-s\|^2] \\
& + \sum_{i=1}^{k} \lambda_i [(xD_i x)^{1/2} - (sD_i s)^{1/2}] \\
& \text{(by the } \rho - \text{convexity assumptions)} \\
\geq & - \sum_{i=1}^{k} \lambda_i [(x-s)D_i w_i + \sum_{i=1}^{k} \lambda_i [(xD_i x)^{1/2} - (sD_i s)^{1/2}] \\
& + [\sum_{i=1}^{k} (\lambda_i \rho_i - \lambda_i y_i \rho_i^*) + \sum_{j=1}^{m} v_j \rho_j^{**}] \|x-s\|^2 \\
& \text{(by (7), (9), (10) and Lemma 2.1)} \\
\geq & [\sum_{i=1}^{k} (\lambda_i \rho_i - \lambda_i y_i \rho_i^*) + \sum_{j=1}^{m} v_j \rho_j^{**}] \|x-s\|^2 \\
\geq & 0.
\end{aligned}
$$

This is a contradiction. Hence the result holds.

Theorem 4.2. Suppose that x^0 is a properly efficient solution of (P) and the set Z^0 is empty. Then there exists a feasible solution $(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ of (D) for some $\lambda > 0$. Furthermore suppose that f_i is ρ_i-convex, $-h_i$ is ρ_i^*-convex, $i = 1, 2, \cdots, k$, and g_i is ρ_j^{**}-convex, $j = 1, 2, \cdots, m$ and that

$$\sum_{i=1}^{k}(\lambda_i \rho_i - \lambda_i y_i^0 \rho_i^*) + \sum_{j=1}^{m} v_j^0 \rho_j^{**} \geq 0.$$

Then $(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ is a properly efficient solution of (D) and their respective extreme values are equal.

Proof. By Theorem 2.1, $(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ is feasible for (D). By Theorem 4.1, their respective extreme values are equal. Following Theorem 4 in [15], $(x^0, v^0, y^0, w_1^0, \cdots, w_k^0)$ is a properly efficient solution of (D).

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