The Moment - Generating Function and Negative Integer Moments

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1. INTRODUCTION

The moment-generating function of a random variable X is defined as $M_X(t) = E(e^{tx})$, provided the expectation $E(\,)$ exits in an interval $|t| < h$.furthermore, $\lim_{t \to 0-} (d^n M_X(t)/dt^n) = E(X^N)$. We will now show that the mgf also generates negative moments, provided certain regularity conditions are met. Williams (1941) is the earlist appearance of this type of result we could find. Analogously, Chao and Strawdermans (1972) have used the probability - generating function to find the negative integer moments of $X + A > 0$, where X is a random variable and A a constant ; see also Kabe (1976) and Schuh (1981) for an application in a branching process problem.

2. THE RESULTS

Suppose for the moment that X is a positive random variable.

Since $x \equiv \int_{-\infty}^{0} e^{t/x}\,dt (x > 0)$,

$$\begin{aligned} E(X) &= \int_0^\infty x dF(x) = \int_0^\infty \int_{-\infty}^{0} e^{t/x}\,dt dF(x) \\ &= \int_{-\infty}^{0} \int_0^\infty e^{t/x}\,dF(x) = \int_{-\infty}^{0} M_{x-1}(t) dt \\ &= \int_0^\infty M^{X-1}(-t) dt. \end{aligned}$$

The interchange of this order of intergration is subject to $E(e^{-t/x})$ being integrable from $t=0$ to $t=\infty$.

Finally, by substituing X^{-1} for X, we find

$$E(X^{-1}) = \int_0^\infty M_X(-t)\, dt, \tag{1}$$

if either integral exists. Performing the integration analytically may not be easy; how-ever (1) dose give an alternative way of evaluating an inverse moment (perhaps even numerically).

There are two natural ways to generalize (1) to $E(X^{-n})$; one way gives

$$E(X^{-n}) = \int_0^\infty \int_{t^1}^\infty \cdots \int_{t_{n-1}}^\infty M_X(-t_n)\, dt_n \cdots dt_2\, dt_1 \tag{2}$$

while the second way gives

$$E(x^{-n}) = \Gamma(n)^{-1} \int t^{n-1} M_X(-t)\, dt\, , \tag{3}$$

Probably the most important extension form (1) is

$$E(Y/X) = \int_0^\infty \lim_{t_2 \to 0-} (\partial/\partial t_2) M_{X,Y}(-t_1, t_2)\, dt_1\, , \tag{4}$$

if either integral exits, where $M_{X,Y}(t_1, t_2) = E(e^{t_1x+t_2y})$ is the joint mgf of X>0 and Y. Equation (4) can be very useful, since ratio statistics and equations concering their bias arise frequently in statistical analyses. For example, Williams(1941) looked at moments of the ratio of the mean squared successive difference to the squared differ-ence from a nomal population, using a variant of (4). S-imilarly, Halperin and Gurian (1971) caculated bias and mean squared error for the usual least squares slope estimator when both variables are subjet to error.

Now relax the assumption that X be a positive random variable, although the restriction that $F(0+)=F(0)$ is necessary to avoid degeneracy: define $sgn(x)=1$ if $x \geq 0$, $=-1$ if $x<0$. Then $X^{-1} = Y/|X|$ almost surely, where $Y = sgn(X)$. Thus $E(X^{-1})$ is given by (4), after calculating $M_{|X|,Y}(t_1, t_2)$.

The interpretation of (1) to (4) deserves some comment. These equations are merely expressing the well-known duality between function space and transfom space. Negative moments clearly pertain to the behavior of the distribution at the origin, which in turn suggests something about the behavior of the transform at infinity. Also, there is the pkesing sysmmetry that whereas positive moments are generated by successive differentiations of the mgf, negative moments are aconsequence of suces -sive integrations. The next section expands a little on this.

3. INVERSE MOMENTS AND LAPLACE AND MELLIN TRNSFORMS

Suppose (in the terminology of Abramowitz and Stegun 1965) that the original function is $G(t)$ and the image function is $g(s)=\int_0^\infty e^{-st}t^{-1}G(t)dt$. Then in Abram-owitz and Stegun (1965, p. 1021), for example, we see that if the original function is $t^{-1}G(t)$,its Laplace transform is $\int_0^\infty g(x)dx$; that is

$$\int_s^\infty g(x)=\int_s^\infty e^{-st}t^{-1}G(t)dt.$$

If we put $s=0$, and interpret $G(t)$ as the density function of a positive random variable X with mgf $M_X(s)=g(-s)$, we then have exactly (1). Thus the result is by no means new, although most statisticians have probably not been aware of it.

The Mellin transform $h(z)=\int_0^\infty H(x)x^{z-1}dx$ of the function H(x) is a function of the (complex) parameter z. If we interpret H as adensity function of a positive random variable X, then knowledge of the Mellin transform tells us all moments of X, positive integer, negative integer, fractional, and so forth. This is hint then that all moments are probably obtainable from the mgf by generalizing differentiation to fr-actional diffentitation, including integrating as a special case of negative integer diffe-rentiation. Thus the αth moment can be obtained from the αth fractional derivative of the mgf, $\alpha \in R$ (see, e.g., Oldham and Spanier 1974). Indeed, Laue (1980) has considered this idea for characteristic functions ; fractional derivatives are used for the formulation of new conditions on the existence of positive real moments of non-nega-tive random variables. We will not pursue this matter here, since we believe that it detracts from the simplicity and thrust of (1), (2), (3),and (4).

4. EXAMPLE

Example 1. The inverse moment of $aX+b$ is easily found by using (1) :

$$E((aX+b)^{-1})=\int_0^\infty e^{-bt}M_X(-at)dt\quad .$$

Example 2. Suppose X is gamma distributed with scale parameter $\alpha>0$ and shape parameter $\lambda>0$.
Then

$$M_X(t)=(1-\alpha t)^{-\lambda}.$$

Therefore, from (1),

$$E(X^{-1})=\alpha^{-1}(\lambda-1)^{-1},$$

provided ($\lambda>1$), and from formulas (2), (3),

$$\mathrm{var}(X^{-1}) = \alpha^{-2}(\lambda-1)^{-2}(\lambda-2)^{-1} \quad ,\lambda > 2$$

Example 3. Consider the important problem of estimating the success probability for a negative binomial distribution. In this familiar situation, we let N denote the random number of trials required to obtain a fixed number r , of successes. Let p be the probability of a success on any trial. Then

$$M_N(t) = [pe^{t}/(1-qe^t)]^r, \quad \infty < t < -\log q,$$

where $q = 1-p$. The maximum likelihood estimator $\hat{p}$ of p is r/N, whose expectation is

$$E(\hat{p}) = rE(N^{-1}) = r\int_0^\infty p^r/(e^t - q)^r dt\ ,$$

using (1). Putting $u = 1-qe^{-t}$ and then expanding $(1-u)^{r-1}$ by using the binomial theorem gives

$$E(P) = r(-1)^R (p/q)[\log p - \sum_{s=1}^{r-1} (-1)^s \binom{r-1}{s} (1-p^s)/sp^s].$$

Example 4. A beta random variable B, with parameters ($\lambda > 0$),($\mu > 0$), can be written as $B = U/(U+V)$, where U, V are independent gamma random variables with parameters (α), (λ), and (α), (μ), respectively ($\alpha > 0$). Then

$$M_{U+V,U}(t_1, t_2) = (1-\alpha t_1)^{-\mu} \cdot (1-\alpha(t_1+t_2))^{-\lambda}.$$

Partial differentiation with respect to t_2 and integration with respect to t_1 yields, using (4), $E(b) = \lambda/(\lambda+\mu)$ which is a well-known result.

5. CONCLUSION

Sometimes a solution distribution can only be written in terms of its mgf, the inversion being too difficult. In particular, the use of mgf's arises when independent random variables are being added. Although the distribution might be inaccessible (positive), moments are easily derived from differentiation. This article shows that negative moments are also hidden in the mgf. Indeed, the most general result we can present is for $X > 0$, Y random variables with joint mgf $M_{X,Y}(t_1, t_2)$. Then it can be easily seen that (1),(3),and(4) are special cases of

$$E(Y^j/X^k) = \Gamma(k)^{-1}\int_0^\infty t_1^{k-1} \lim_{t_2 \to 0-} \partial^j M_{X,Y}(-t_1, t_2) / \partial t_2^j\, dt_1$$

where j=0, 1, 2, . . . , k = 1, 2, 3, . . . , and when either integral exists. The result has been in the literature in various forms for some time, but it is certainly not well known.

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