# TESTING MEANS USING HYPOTHESIS–DEPENDENT VARIANCE ESTIMATES

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## 1. Introduction

While introducting the concepts of hypothesis trsting to beginning statistics students, the analogy between the process of jury trial of an accused and the process of hypothesis testing is often drawn. The hypothesis testing approach is also compared with the approach of "proof by contraction". The students are repeatedly reminded about the logic of proceeding under the assumption of the null hypothesis being true unless contradicted through the sample evidence(at some specified level of significance).

As various hypothesis testing procedures are introduced, some students become puzzled by the fact that the information specied in the null hypothesis is not fully utilized in some of the tests, especially in computing the variance estimates. For example, under the one sample $t$ tests, the sample mean, $\overline{x}$, is utilized (instead of the specified mean, $\mu_0$)in computing the variance estimate, while in testing for the binomial proportion, the specified proportion value, $p_0$, is utilized in computing the variance estimate.

This article examines some of the common univariate and multivariate tests from this view point. It turns out that in all but one test considered, the use of hypothesis-dependent variance estimates leads to tests that are equivalent to the correspondending traditional tests utilizing the hypothesis-independent variance estimates. The realization of this fact mat be interesting and educational to practitioners and students of ststistics.

## 2. Univariate Tests

The one sample $t$ test is commonly used to test the null hypothesis

$$H_0 : \mu = \mu_0$$

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based on a random sample from a normal distribution with unknown variance $\sigma^2$. This test is the likelihood ratio test as well as the uniformly most powerful unbiased test. The test statistic $t$ is given by

$$t = (\bar{x} - \mu_0)/(s^2/n)^{1/2},$$

where $\bar{x}$ is the sample average based on a random sample of size $n$. The sample variance $s^2$ is an unbiased estimate of $\sigma^2$ computed as

$$\widehat{\sigma}^2 = s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1).$$

The question of interest is why don't we take advantage of the specified value of $\mu$ under the null hypothesis in computing as estimate of $\sigma^2$, as

$$\widehat{\sigma}^2 = s_0^2 = \sum_{i=1}^{n}(x_i - \mu_0)^2/n?$$

In turn, why don't we use $t_0$ as our test statistic to test the hypothesis in (2.1) where

$$t_0 = (\bar{x} - \mu_0)^2/(s_0^2/n)^{1/2}?$$

The answers to these questions are rather simple. It can be easily shown that the $t_0$ statistic can be written in terms ofthe $t$ statistic as follows:

$$t_0 = t[n/(n-1+t^2)]^{1/2}. \tag{2.1}$$

The one-to-one correspondence between the statistics $t$ and $t_0$ and their corresponding distributions(percentile points) are now apparent. Note that even though $\bar{x}$ and $s_0^2$ are dependent, the distribution of $t_0$ can still be easily obtained from the distribution of $t$ through (2.1). In summary, the $t$ and $t_0$ tests lead to the same inference and hence are identical tests. It is interesting to note that a gain of one degree of freedom really dose not improve the test. So why bother with the $t_0$ test? For this and related discussions, see Lefante and Shah(1986) and Good(1986). In the same spirit, one can raise a similar question about the $F$ statistic for testing the equality of the $k$ teratment means, that is,

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_k,$$

in a completely randomized design. The same logic leads to a new statistic, $F_0$, which can be expressed in terms of the usual $F$ statistic as shown below:

$$F_0 = \frac{\text{treatment mean squares}}{\text{total mean squares}} = \frac{(N-1)F}{(k-1)F+(N-k)}.$$

Once again, the one-to-one correspondence between the statistics $F$ and $F_0$ for the completely randomized design and their corresponding distributions lead to the same inference and hence are identical tests. So there is no need to bother with the $F_0$ test.

### 3. Multivariate tests.

Noe we continue with this same idea in the one-sample and $k$-sample multivariate testing procedures. Suppose that we have independent vectors $X_1, \ldots, X_n$ from a $p$-variate normal population with unknown mean vector $\xi$ and unknown covariance matrix $\sum$. The usual test statistic for testing

$$H_0 : \xi = \xi_0$$

is the Hotelling's $T^2$, which is given by

$$T^2 = n(\overline{X} - \xi_0)' S^{-1} (\overline{X} - \xi_0), \tag{3.1}$$

where $X$ is the sample mean vector and

$$S = \sum_{i=1}^{n} (X_i - \overline{X})(X - \overline{X})'/(n-1)$$

is the sample covariance matrix. If we replace $S$ in (3.1) by the hypothesis-dependent covariance estimate,

$$S_0 = \sum_{i=1}^{n} (X_i - \xi_0)(X_i - \xi_0)'/n,$$

do we get a different test? That is, dose

$$T_0^2 = n(\overline{X} - \xi_0)' S_0^{-1} (\overline{X} - \xi_0)$$

lead to a different test? It is easy to show that $T_0^2$ can be expressed as

$$T_0^2 = T^2[n/(n-1) + T^2],$$

which is a one-to-one function of $T^2$.

Thus, for the same reasons given in the $t$ test case, we conclude that the test procedures based on $T^2$ and $T_0^2$ are identical. Interestingly, Kshirsager(1972, problem 40, p.490) has noted a statistic $T'^2$ which equals $(n-1)T_0^2/n$.

Finally, we investigate the effect of hypothesis-dependent covariance estimates in the **MANOVA** setup for testing the equality of several normal mean vector and the unknown covariance matrix of $i$th population, $i = 1, \ldots, k$. Further let the $j$th observation from the $i$th population, $i = 1, \ldots, k$ and $j = 1, 2, \ldots, n_i$ be denoted by $X_{ij}$ and the $j$th sample mean vector by $\overline{X}_i$.

Define the hypothesis matrix

$$H = \sum_{i=1}^{n} n_i(\overline{X}_i - \overline{X})(\overline{X}_i - \overline{X})',$$

where $\overline{X} = \sum_{i=1}^{k} n_i\overline{X}_i/(\sum_{i=1}^{k} n_i)$, and the error matrix

$$E = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)(X_{ij} - \overline{X}_i)'.$$

then, the usual test statistics to test

$$H_0 : \xi_0 = \cdots = \xi_k$$

are some real valued functions of the eigenvalues of $HE^{-1}$(Morrison 1990). For example, Roy's root test statistic is $\theta_s = c_s/(1+c_s)$, where $c_s$ is the largest eigenvalue of $HE^{-1}$ among the $s$ nonzero eigenvalues of $HE^{-1}$. The Lawley-Hotelling test statistic is tr$(HE^{-1}$. Again, the question of interest is whether we get different test procedures if we use the hypothesis-dependent error matrix

$$E_0 = \sum_{i=0}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X})(X_{ij} - \overline{X})'$$

instead of $E$. We show that two commonly used real valued functions of the eigenvalue of $HE_0^{-1}$, namely the largest eigenvalue and the trace of $HE_0^{-1}$, lead to the well-known test procedures available in the literature.

First let us consider the largest eigenvalue of $HE_0^{-1}$ as a test statistic to test $H_0$. Noting the relation that $E_0 = H + E$, it can be easily shown that the largest eigenvalue of $HE_0^{-1}$ is indeed Roy's root test statistic $\theta_s$. If we propose $\mathrm{tr}(HE_0^{-1})$ as a test statistic, then this leads to a different test than the Lawley-Hotelling test.
However, this is the well-known Pillai's trace test statistic, as $\mathrm{tr}(HE_0^{-1}) = \mathrm{tr}[H(H+E)^{-1}]$.

## References

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