이 논문은 국비 해외 교수 파견(1983. 11. 21~1984. 11. 21) 연구비에 의해 수행되었음.

Menger 空間에 對한 不動点 定理

金　　章　　郁

Fixed Point Theorems on Menger Spaces

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〈Contents〉

要　約

Menger空間은 거리공간의 確率的 意味에서의 擴張이며, 本 論文의 目的은 Menger空間의 位相的 槪念들을 考察하고, 縮小型函數, 縮小型函數條件을 만족하는 두 函數, 縮小型 函數條件을 만족하는 函數의 集合族들에 대한 여러가지 不動點定理를 얻는 것이다.

(0) Introduction

K. Menger [3] introduced the notions of satistical metric spaces which could be considered the generalization of metric spaces, and Sehgal and Bharucha-Reid [7] proved a fixed point theorem on the Menger Space. Recently J. I. Chang and C. W. Kim gave generalized fixed point theorems on Menger spaces.

The aim of this paper is to investigate common fixed point theorms.

We can obtain information from many common fixed point theorem on metric spaces on normed spaces which were studied by many authors. In section 1 we introduce the notions of Menger spaces and their topological properties. In section 2, 3, we prove various types of common fixed point theorems on Menger spaces.

(1) Basic definitions

Let $\mathbf{R}$ be the set of reals and $\mathbf{R}^+=\{x\in\mathbf{R}\mid x\geq 0\}$. A mapping $F:\mathbf{R}\to\mathbf{R}^+$ is said to be a distribution function if it is nondecreasing left-continuous with inf $F=0$ and sup $F=1$. The set of all distribution functions will be denoted by L.

Definition 1.1. A statistical metric space (SM-space) is an ordered pair $(S,\mathscr{F})$, where S is a (nonempty) set and $\mathscr{F}$ is a mapping of $S\times S$ into L (we shall denote $\mathscr{F}(p,q)$ by F_{pq}) satisfying

(SM-I) $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$

(SM-II) $F_{pq}(0)=0$

(SM-III) $F_{pq}=F_{qp}$

(SM-IV) if $F_{pq}(x)=1$ and $F_{qr}(y)=1$, than $F_{pr}(x+y)=1$,

for p, q, $r\in S$ and x, $y\in R$.

Definition 1.1 suggests that $F_{pq}(x)$ may be interpreted as the probability that the distance between p and q is less than x.

Definition 1.2. A Menger space is a triple $(S,\mathscr{F},\Delta)$, where $(S,\mathscr{F})$ is a SM-space and $\Delta:[0,1]\times[0,1]\to[0,1]$ is a mapping satisfying

(Δ-I) $\Delta(a,1)=a$, $\Delta(0,0)=0$

(Δ-II) $\Delta(a,b)=\Delta(b,a)$

(Δ-III) $\Delta(c,d)\geq\Delta(a,b)$ if $c\geq a$ and $d\geq b$

(Δ-IV) $\Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$

(Δ-V) $F_{pr}(x+y)\geq\Delta(F_{pq}(x),F_{qr}(y))$

for all $p,q,r\in S$ and for all $x\geq 0$, $y\geq 0$.

A mapping $\Delta:[0,1]\times[0,1]\to[0,1]$ satisfying (Δ-1)$-$(Δ-IV) is said to be a Δ-norm.

The topology of a SM-space was introduced by Schweizer and Skla [5]. Let $p\in S$, and ε,λ be positive reals. Then an (ε,λ)-neighborhood of p is defined by

$$N_p(\varepsilon,\lambda)=\{q\in S\mid F_{pq}(\varepsilon)>1-\lambda\}.$$

Due to [5] and [6], if $[(S,\mathscr{F},\Delta)$ is a Menger space, and Δ is continuous, then it is a Hausdorff space satisfying the first axiom of countability induced by the family $\{N_p(\varepsilon,\lambda)\mid p\in S,\ \varepsilon>0,\lambda>0\}$ of neighborhoods.

Definition 1.3. Let $(S,\mathscr{F},\Delta)$ be a Menger space. A sequence $\{p_n\}$ in S is said to be fundamental in S if for each $\varepsilon>0$, $\lambda>0$, there exists an integer $M(\varepsilon,\lambda)$, such that $F_{p_np_m}(\varepsilon)>1-\lambda$ whenever $n,m\geq M(\varepsilon,\lambda)$.

A Menger space S is complete if each fundamental sequence in S converges to an element in S.

The following theorem establishes a connection between metric spaces and Menger spaces.

Theorem 1.1. [7]. *If (S,d) is a metric space, then the metric d induces a mapping $\mathscr{F}: S\times S\to L$, where $\mathscr{F}(p,q)$ $(p,q\in S)$ is defined by $\mathscr{F}(p,q)(x)=H(x-d(p,q))$, $x\in\mathbf{R}$, where $H(x)=0$ if $x\leq 0$ and $H(x)=1$ if $x>0$. Further, if $\Delta:[0,1]\times[0,1]\to[0,1]$ is defined by $\Delta(a,b)=\min\{a,b\}$, then $(S,\mathscr{F},\Delta)$ is a Menger space. It is complete if the metric d is complete.*

The space $(S,\mathscr{F},\Delta)$ so obtained is called the induced Menger space.

(2) Common Fixed Point Theorems for Two Mappings

Definition 2.1. Let $(S,\mathscr{F})$ be an SM-space. Let : R→R be an upper semicontinuous mapping from the right such that $\phi(x)=0$ if $x\leq 0$ and $0<\phi(x)<x$ for all $x>0$. Two mappings T_1 and T_2 of S into itself are said to satisfy the CT-mapping condition if for all p,q in S and for all $x>0$, $F_{T_1pT_2q}(\phi(x))\geq F_{pq}(x)$.

Theorem 2.1. Let $(S,\mathscr{F},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. If $T_1$and T_2 are mappings of S into itself satisfying the CT-mapping condition, then T_1 and T_2 have a unique common fixed point p in S.

Proof. Consider a point p_o in S and define a sequence $\{p_n\}$ by $p_1=T_1p_o$, $p_2=T_2p_1$, $p_3=T_1p_2$, $\cdots$, $p_{2n}=T_2p_{2n-1}$, $p_{2n+1}=T_1p_{2n}$, $\cdots$. Since T_1 and T_2 satisfy the CT-mapping condition, we have $F_{p_{2n+1}p_{2n+2}}(\phi(x))=F_{T_1p_{2n}T_2p_{2n+1}}(\phi(x))\geq F_{p_{2n}p_{2n+1}}(x)$, for all $x>0$. Similarly, $F_{p_{2n}p_{2n+1}}$

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(2) Common Fixed Point Theorems for Two Mappings

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Theorem 2.1. Let $(S,\mathscr{F},\varDelta)$ be a complete Menger space and $\varDelta$ be a continuous function satisfying $\varDelta(x,x)\geq x$ for each $x\in[0,1]$. If $T_1$and T_2 are mappings of S into itself satisfying the CT-mapping condition, then T_1 and T_2 have a unique common fixed point p in S.

Proof. Consider a point p_o in S and define a sequence $\{p_n\}$ by $p_1=T_1p_o$, $p_2=T_2p_1$, $p_3=T_1p_2$, $\cdots$, $p_{2n}=T_2p_{2n-1}$, $p_{2n+1}=T_1p_{2n}$, $\cdots$. Since T_1 and T_2 satisfy the CT-mapping condition, we have $F_{p_{2n+1}p_{2n+2}}(\phi(x))=F_{T_1p_{2n}T_2p_{2n+1}}(\phi(x))\geq F_{p_{2n}p_{2n+1}}(x)$, for all $x>0$. Similarly, $F_{p_{2n}p_{2n+1}}(\phi(x))\geq F_{p_{2-1}p_{2n}}(x)$, for all $x>0$. Hence, $F_{p_np_{n+1}}(\phi(x))\geq F_{p_{n-1}p_n}(x)$, for all $x>0$ and $n=1$, 2, $\cdots$. Now, we claim that the sequence $\{p_n\}$ is a Cauchy sequence in S. Suppose not. Then there exist an $\varepsilon>0$ and $\lambda>0$ such that for any positive integer 1, we can find n, m with $n>m\geq 1$, n:an odd integer m:an even integer and

$$F_{p_np_m}(\phi(\varepsilon))<1-\lambda,\ F_{p_{n-1}p_m}(\phi(\varepsilon)\geq 1-\lambda.$$

For this, we can choose an $x>0$ such that $F_{p_op_1}(x)\geq 1-\lambda$. Since $\lim\phi^n(x)=0$, we may choose a positive integer N with $\phi^N(x)\leq\phi(\varepsilon)$. Since $F_{p_np_{n+1}}(\phi(x))\geq F_{p_{n-1}p_n}(x)$ for all $x>0$, and $n=1,2,\cdots$, we have $F_{p_np_{n+1}}(x)\geq F_{p_np_{n+1}}(\phi(x))\geq F_{p_np_{n+1}}(\phi^n(x))\geq F_{p_op_1}(x)\geq 1-\lambda$, for all $n\geq N$. Suppose that 1 is sufficiently large such that $1\geq N+1$ and $\varepsilon-\phi(\varepsilon)\geq\phi^1(x)$. Since $F_{p_{n-1}p_n}(\phi(\varepsilon))\geq 1-\lambda$, we have

$$F_{p_{n-1}p_{m-1}}(\varepsilon) \geq \Delta(F_{p_{n-1}p_m}(\phi(\varepsilon)), F_{p_m p_{m-1}}(\varepsilon-\phi(\varepsilon))$$
$$\geq \Delta(1-\lambda,\ 1-\lambda)$$
$$\geq 1-\lambda.$$

Since T_1 and T_2 satisfy the CT-mapping condition,

$$F_{p_n p_m}(\phi(\varepsilon)) = F_{T_1 p_{n-1} T_2 p_{m-1}}(\phi(\varepsilon)) \geq F_{p_{n-1}p_{m-1}}(\varepsilon) \geq 1-\lambda.$$

Since $F_{p_n p_m}(\phi(\varepsilon)) < 1-\lambda$, $1-\lambda > 1-\lambda$. This is a contradiction. Thus $\{p_n\}$ is a Cauchy sequence in S.

Since $(S, \mathscr{F}, \Delta)$ is complete, $\{p_n\}$ converges to a point p in S. Now, we shall prove that $T_{1p}=p$, $T_{2p}=p$.

For all $x>0$, we have

$$F_{T_{1p}p_{2n+2}}(x) = F_{T_{1p}T_{2p_{2n+1}}}(\phi(x)) \geq F_{p p_{2n+1}}(x)$$

we have

$$F_{T_{1p}p}(x) = \liminf F_{T_{1p}p_{2n+2}}(x)$$
$$\geq \liminf F_{p p_{2n+1}}(x)$$
$$= F_{pp}(x)$$
$$=1.$$

Thus, $T_{1p}=p$. Similarly, $T_{2p}=p$.

Now suppose that q is another common fixed point of T_1 and T_2 in S. Then,

$$F_{pq}(\phi(x)) = F_{T_{1p}T_{2q}}(\phi(x)) \geq F_{pq}(x) \text{ for all } x>0.$$

Similarly, $F_{pq}(\phi^2(x)) \geq F_{pq}(\phi(x))$. Hence was have

$$F_{pq}(\phi^n(x)) \geq F_{pq}(x) \text{ for all } x>0 \text{ and for } n=1,\ 2, \cdots.$$

Since $\lim \phi^n(x)=0$, for any $\varepsilon>0$ and $\lambda>0$ there exist an $x>0$ ann a positive integer n such that $F_{pq}(x)>1-\lambda$ and $\phi^n(x) \leq \varepsilon$. Therefore, we have $F_{pq}(x)>1-\lambda$, so that $F_{pq}(x)=1$ for all $x>0$. Hence $p=q$.

As immediate corollaries, we have the followings

Corollary 2.1 [2]. Let (S, d) be a complete metric space and ϕ be as in definition 3.1.1. If T_1 and T_2 are mappings of S into itself such that $d(T_{1p}, T_{2q}) \leq \phi(d(p,q))$ for all all p, q in S, then T_1 and T_2 have a unique common fixed point p in S.

Corollary 2.2. Let $(S, \mathscr{F}, \Delta)$ be a complete Menger space and be a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$. If T_1 and T_2 are mappings of S into itself such that there exists a constant k, $0<k<1$ such that for all p, q in S.

$F_{T_{1p}T_{2p}}(kx) \geq F_{pq}(x)$ for all $x<0$, then T_1 and T_2 have a unique common fixed point y in S.

We can consider another type of a common fixed point theorem in a complete Menger space.

Theorem 2.2. Let $(S, \mathscr{F}, \Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$. Let ϕ be as in definition 2.1. If T_1 and T_2 are mappings of S into itself satisfying

(1) $F_{T_1T_{2p}T_2T_{1q}}(\phi(x)) \geq F_{pq}(x)$ for all p, q in S and for all $x>0$, then T_1 and T_2 have a unique common fixed point p in S.

Proof. Let $U=T_1T_2$, $V=T_2T_1$. Then by theorem 2. 1, we have a unique common fixed point p of U and V.

$$T_{1p}=T_1V_p=T_1(T_2T_{1p})=T_1T_2(T_{1p})=UT_{1p}.$$
$$T_{2p}=T_2U_p=T_2(T_1T_{2p})=T_2T_1(T_{2p})=VT_{2p}.$$

Hence T_{1p}, T_{2p} are fixed points of U, V respectively.

By (1), $F_{T_{1p}T_{2p}}(\phi(x))=F_{UT_{1p}VT_{2p}}(\phi(x)) \geq F_{T_{1p}T_{2p}}(x)$ for all $x>0$.

Similarly, $F_{T_{1p}T_{2p}}(\phi^2(x)) \geq F_{T_{1p}T_{2p}}(x)$. Hence, we have

$F_{T_{1p}T_{2p}}(\phi^n(x)) \geq F_{T_{1p}T_{2p}}(x)$ for all $x>0$ and for $n=1, 2, \cdots$.

By the similar method of theorem 2.1, we can prove that $T_{1p}=T_{2p}$.

By the uniqueness of the common fixed point of U and V, we have $p=T_{1p}=T_{2p}$. Theorefore p is a common fixed point of T_1 and T_2.

It is clear that the common fixed point of T_1 and T_2 is unique.

Corollary 2.3. Let (S, d) be a complete metric space and ϕ be as definition 2.1. If T_1 and T_2 are mappings of S into itself satisfying the following condition:

$d(T_1T_{2p}, T_2T_{1q}) \leq \phi(d(p,q))$ for all p, q in S, then T_1 and T_2 have a unique fixed point p in S.

Corollary 2.4. Let $(S, \mathscr{F}, \Delta)$ be a complete Menger space and be a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$.

If T_1 and T_2 are mappings of S into itself satisfying the following condition: there exists a constant k, $0<k<1$, such that

$F_{T_1T_{2p}T_2T_{1q}}(kx) \geq F_{pq}(x)$ for all p, q in S and for all $x>0$, then T_1 and T_2 have a unique common fixed point p in S.

A fixed point of a selfmapping T of S can be considered as a common fixed point of T and I, the identity mapping of S. In certain cases, we can replace I by a continuous selfmapping U of S and consider common fixed points of U and T.

Theorem 2.3. Let $(S, \mathscr{F}, \Delta)$ be a complete Menger space and Δ be a continunus function satifying $\Delta(x,x) \geq x$ for each $x \in [0,1]$. Let ϕ be as in definition 2.1. Let U be a continuous mapping of S into itself and T be a mapping of S into itself commuting with U such that $F_{TpTq}(\phi(x)) \geq F_{UpUq}(x)$, for all p, q in S and for all $x>0$, If there exists a point q_0 in S such that we can choose a sequence $\{Uq_n\}$ recursively given by $Uq_n = Tq_{n-1}$ for $n=1,2,\cdots$, then U and T have a unique common fixed point in S.

Proof. $F_{Uq_{n+1}Uq_{n+}}(\phi(x)) = F_{Tq_nTq_{n-1}}(\phi(x)) \geq F_{Uq_nUq_{n-1}}(x)$ for all $x>0$.

Now, we claim that $\{Uq_n\}$ is a Cauchy sequence in S. Suppose not. Then there exist an $\varepsilon>0$, and a $\lambda>0$, for any positive integer 1, we can find n, m with $n>m\geq 1$ such that

$$F_{Uq_nUq_m}(\phi(\varepsilon)) < 1-\lambda, \quad F_{Uq_{n-1}Uq_m}(\phi(\varepsilon)) \geq 1-\lambda.$$

For this, we can choose an $x>0$ such that $F_{Uq_1Uq_2}(x) \geq 1-\lambda$.

Since $\lim_{n\to\infty} \phi^n(x)=0$, we may choose a positive integer N with $\phi^{N+1}(x) \leq \phi(\varepsilon)$, $F_{Uq_{n+1}Uq_n}(\phi(\varepsilon)) \geq F_{Uq_1Uq_2}(x) \geq 1-\lambda$ for all $n \geq N$.

Suppose 1 is sufficiently large such that $1 \geq N+2$ and $\varepsilon - \phi(\varepsilon) \leq \phi^{1+2}(x)$. Then we have

$$F_{Uq_{n-1}Uq_n}(\varepsilon) \geq F_{Uq_{n-1}Uq_n}(\phi(\varepsilon)) \geq F_{Uq_1Uq_2}(x) \geq 1-\lambda \text{ and}$$

$$F_{Uq_{n-1}Uq_{m-1}}(\varepsilon) \geq \Delta(F_{Uq_{n-1}Uq_m}(\phi(\varepsilon)), \; F_{Uq_mUq_{m-1}}(\varepsilon-\phi(\varepsilon)).$$

Therefore, we have $1-\lambda > 1-\lambda$. This is a contradiction. Therefore $\{Uq_n\}$ is a Cauchy sequence in S.

Since $(S, \mathscr{F}, \Delta)$ is complete, there exists a point p in S such that Uq_n converges to p. Since U is continuous,

$\{UUq_n\} = \{UTq_{n-1}\} = \{TUq_{n-1}\}$ converges to Up.

For all $x>0$,

$$\begin{aligned} F_{TpUp}(\phi(x)) &= \lim\inf F_{TpTUq_{n-1}}(\phi(x)) \\ &\geq \lim\inf F_{UpU^2q_{n-1}}(x) \\ &= F_{UpUp} \\ &= 1. \end{aligned}$$

Thns Tp=Up.

Now we prove that $Up=UUp=TUp$. For this, put $q=Tp=Up$.

For all $x>0$, we have

$$F_{qUq}(\phi(x)) = F_{TpUTp}(\phi(x)) = F_{TpTUp}(\phi(x)) \geq F_{UpU^2p}(x) = F_{qUq}(x).$$

Hence $F_{qUq}(\phi^n(x)) = F_{qUq}(x)$ for all $x>0$ and $n=1,2,\cdots$. By the similar method of theorem 3.1, $q=Uq$.

$Tq=TUp=UTq=Uq=q$. Hence Up is a common fixed point of U and T. Let q' be a common fixed point of U and T. Then for all $x>0$, we have $F_{qq'}(\phi(x)) = F_{TqTq'}(\phi(x))$ $F_{UqUq'}(x) = F_{qq'}(x)$.

Hence $F_{qq'}(\phi^n(x) \geq F_{qq}(x)$ for all $x>0$ and for $n=1,2, \cdots$.

By the similar method of theorem 2.1, $q=q'$.

As immediate corollaries, we have the followings.

Corollary 2.5. Let $(S,\mathscr{F},\Delta)$ be a complete Menger space and be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$.

Let ϕ be as in definition 2.1. Let U be a continuous mapping of S into itself and T be a mapping of S into itself commuting with U such that $T(S)\subset U(S)$. If T and $U$are mappings such that $F_{TpTq}(\phi(x))\geq F_{UpUq}(x)$ for all p,q in S and for all $x>0$, then U and T have a unique common fixed point in S.

Proof. Since $T(S)\subset U(S)$, for any $p\in S$, we can choose a sequence $\{Uq_n\}$ such that $Uq_n=Tq_{n-1}$ for $n=1,2,\cdots$. Therefore, by theorem 2.3, we have the above result.

Corollary 2.6. Let (S,d) be a complete metric space and ϕ be as definition 2.1. Let U be a continuous mapping of S into itself and T be a mapping of S into itself commuting with U such that $d(Tp,Tp)\leq\phi(d(Up,Uq))$ for all p,q in S. If there exists a point q_o in S such that we can choose a sequence $\{Uq_n\}$ recursively given by $Uq_n=Tq_{n-1}$ for $n=1,2,\cdots$, then T and U have a unique common fixed point in S.

Corollary 2.7. Let $(S,\mathscr{F},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. Let U be a continuous mapping of S into itself and T be a mapping of S into itself commuting with U such that there exists a constant k, $0<k<1$ such that $F_{TpTq}(kx)\geq F_{UpUp}(x)$ for all p,q in S and for all $x>0$. If there exists a point q_o in S such that we can choose a sequence $\{Uq_n\}$ recursively given by $Uq_n=Tq_{n-1}$ for $n=1,2,\cdots$, then T and U have a unique common fixed point in S.

If the condition $\Delta(x,x)\geq x$ and its completeness of the Menger space are omitted, we have other types of a common fixed point theorem as follows.

Theorem 2.4. Let $(S,\mathscr{F},\Delta)$ be a Menger space and Δ be a continuous function. Let T_1 and T_2 be mappings of S into itself satisfying the CT-mapping conditinn and conditinuous at a point p_o in S If there exists a point q_o in S such that the sequence $p_1=T_1q_o$, $p_2=T_2p_1$, $p_3=T_1p_2$, $\cdots$, $p_{2n}=T_2p_{2n-1}$, $p_{2n+1}=T_1p_{2n}$, $\cdots$, has a subsequence converging to p_o, then p_o is a unique common fixed point of T_1 and T_2.

Proof. Let $\{p_{n_i}\}$ be a convergent subsequence of $\{p_n\}$ which converges to p_o. Then the set of indices $\{n_i\}$ contains infinitely many even or odd. Suppose that $\{n_i\}$ contains infinitely many even.

These indices are written in the form of $\{2m_i\}$. Therefore the sequence $\{p_{2m_i}\}$ converges to p_o. Suppose that $p_o \neq T_1 p_o$. Then Suppose that $p_o \neq T_1 p_o$. Then there exists an $y>0$ and an a, 0≤a<1 such that $p_o \neq T_1 p_o$. Then there exists an $y>0$ and an a, $0 \leq a < 1$ such that $F_{p_o T_1 p_o}(y) = a$. Since $\{p_{2m_i}\}$ converges to p_o and T_1 is continuous at p_o, $\{T_1 p_{2m_i}\}$ converges to $T_1 p_o$. For all $x>0$, we have $F_{p_{2m_i} p_{2m_i+1}}(\phi(x)) = F_{T_1 p_{2m_i} T_2 p_{2m_i-1}}(\phi(x)) \geq F F_{p_{2m_i} p_{2m_i-1}}(x)$.

Smilarly, $F_{p_{2m_i} p_{2m_i+1}}(\phi^2(x))$ $F_{p_{2m_i-1} p_{2m_i}}(\phi(x))$.

Hence, we have $F_{p_{2m_i} p_{2mi+1}}(\phi^{2mi}(x)) \geq F_{q_o p_1}(x)$

Find an $x>0$, so that $F_{q_o p_1}(x) > a$ and choose $i_o > 0$ such that $\phi^{2m i_o}(x) \leq y$. Then we have

$$
\begin{aligned}
a = F_{p_o T_1 p_o}(y) &= \lim\inf F_{p_{2mi} p_{2mi+1}}(y) \\
&\geq \lim\inf F_{p_{2m_i} p_{2m_i+1}}(\phi^{2m i_o}(x)) \\
&\geq F_{p_{2mi_o} p_{2mj+1}}(\phi^{2mi_o}(x)) \\
&\geq F_{q_o p_1}(x) \\
&> a.
\end{aligned}
$$

This is a contradiction. Hence we have $T_1 p_o = p_o$.

Since $F_{p_o T_2 p_o}(x) = F_{T_1 p_o T_2 p_o}(x) \geq F_{T_1 p_o T_2 p_o}(\phi(x)) \geq F_{p_o p_o}(x) = 1$ for all $x>0$, we have $T_2 p_o = p_o$.

Now suppose that there exists a point q_o in S such that $p_o' = T_1 p_o'$, $p_o' = T_2 p_o'$. For all $x>0$, $F_{pp}'(\phi^n(x)) \geq F_{pp}'(x)$. By the similar method of theorem 2.1, we have $p = p'$.

As immediate corollaries, we have the followings.

Corollary 2.8. Let (S,d) be a metric space and let T_1 and T_2 be two mappings of S into itself satisfying the condition:

$d(T_1 p, T_2 q) \leq \phi(d(p,q)$ for all p,q in S and T_1 and T_2 are continuous at p_o in S. If there exists a point q_o in S such that the sequence, $p_1 = T_1 q_o$, $p_2 = T_2 p_1$, $p_3 = T_1 p_2, \cdots$, $p_{2n} = T_2 p_{2n-1}$, $p_{2n+1} = T_1 p_{2n}$, $\cdots$, has a subsequence converging to p_o, then p_o is a unique common fixed point of T_1 and T_2.

Corollary 2.9. Let $(S, \mathscr{F}, \Delta)$ be a menger space and Δ be a continuous function. If T_1 and T_2 are mappings of S into itself such that there exists a constant k, $0<k<1$ such that for all p,q in S, $F_{T_1 p T_2 q}(kx) \geq F_{pq}(x)$ for all $x>0$ and T_1, T_2 are continuous at p_o in S. If there exists a point q_o in S such that the sequence $p_1 = T_1 q_o$, $p_2 = T_2 p_1$, $p_3 = T_1 p_2$, $\cdots$, $p_{2n} = T_2 p_{2n-1}$, $p_{2n+1} = T_1 p_{2n}$, $\cdots$, has a subsequence converging to p_o, then p_o is a unique common fixed point of T_1 and T_2.

Theorem 2.5. Let $(S, \mathscr{F}, \Delta)$ be a Menger space and Δ be a continuous function. Let T_1 and T_2 be two mappings S into itself and ϕ be as in definition 2.1. If there exists a point q_o in S such that the sequence $p_1 = T_1 q_o$, $p_2 = T_2 p_1$, $p_3 = T_1 p_2$, $\cdots$, $p_{2n} = T_2 p_{2n-1}$, $p_{2n+1} = T_1 p_{2n}$, $\cdots$, converges to p_o in S and (1) $F_{T_1 p_o T_{2r}}((x)) \geq F_{p_o r}(x)$ for all $x>0$ and for all r

S, then p_o is a unique common fixed point of T_1 and T_2.

Proof. By (1), $F_{T_1p_op_{2n}}(x)=F_{T_1p_oT_2p_{2n-1}}(x)$

$$\geq F_{T_1p_oT_2p_{2n-1}}(\phi(x))$$

$$\geq F_{p_op_{2n-1}}(x), \text{ for all } x>0.$$

By theorem 2.3, $F_{T_1p_op_o}(x)=\liminf_{n\to\infty} F_{T_1p_op p_{2n}}(x)$

$$\geq \liminf_{n\to\infty} F_{p_op_{2n-1}}(x)$$

$$=F_{p_op_o}(x), \text{ for all } x>0.$$

Hence, $F_{T_1p_op_o}(x)=1$. Thus $T_{1p_o}=p_o$.

$F_{T_2p_op_o}(x)=F_{T_1p_oT_2p_o}(x)\geq F_{T_1p_oT_2p_o}(\phi(x))=F_{p_op_o}(x)=1$ for all $x>0$. Hence $T_{2p_o}=p_o$.

By the similar method of theorem 2.1, we can prove the uniqueness of this common fixed point.

From theorem 2.5, we have the followings.

Corollary 2.10. Let (S,d) be a metric space, ϕ be as in definition 2.1 and T_1, T_2 be two mappings of S into itself. If there exists a point q_o in S such that the sequence $p_1=T_1q_o$, $p_2=T_2p_1$, $p_3=T_1p_2$, $\cdots$, $p_{2n}=T_2p_{2n-1}$, $p_{2n+1}=T_1p_{2n}$, $\cdots$, converges to p_o in S and if $d(T_1p_o, T_2r)\leq kd(p_o,r)$ for all r in S and for some k, $0<k<1$, then p_o is a unique common fixed point of T_1 and T_2.

Corollary 2.11. Let $(S,\mathscr{F},\Delta)$, be a Menger space and Δ be a continous function. Let T_1 and T_2 be two mappings of S into itself and ϕ be as in definition 2.1. If there exists a point q_o in S such that the sequence $p_1=T_1q_o$, $p_2=T_2p_1$, $p_3=T_1p_2$,, $p_{2n}=T_2p_{2n-1}$, $p_{2n+1}=T_1p_{2n}$, $\cdots$, converges to p_o in S and if $F_{T_1p_oT_2r}(kx)\geq F_{p_or}(x)$ for all $x>0$, all r in S and some k, $0<k<1$, then p_o is a unique common fixed point of T_1 and T_2.

(3) Common Fixed Point Theorems for the Family of Mappings.

Theorem 3.1. Let $(S,\mathscr{F},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x, x)\geq x$ for each $x\in[0,1]$. If $\{T_i\}$ $(i=1, 2, \cdots, n)$ is a family of mappings of S into itself such that the composition $T_1T_2\cdots T_n$ is a CT-mapping and $T_iT_j=T_jT_i$ $(i,j=1,2, \cdots, n)$, then T_i has a unique common fixed point p in S.

Proof. Let $T=T_1T_2\cdots T_n$. T has a unique fixed point p in S. Thus $Tp=p$. Now for each $i=1,2, \cdots, n$, $T_ip=T_iTp=TT_{ip}$. Hence T_ip is also a fixed point of T. Therefore by the uniqueness, $T_ip=p$ for each $1\leq i\leq n$.

The uniqueness is clear by the uniqueness of the fixed point of T.

As immediate corollaries, we have the followings.

Corollary 3.1. Let (S,d) be a complete metric space and let $\{T_i\}$ $(i=1,2,\cdots,n)$ be a family of mappings of S into itself satisfying the condition for each i, $j(i,j=1,2,\cdots)$ with $i\neq j$, $d(T_{ip}, T_{jq})\leq Kd(p,q)$ for all p,q in s and for some K, $0<K<1$. Then $\{T_i\}$ has a unique common fixed point p in s.

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