Journal of the Research Institute of Basic Sciences,
Korea Maritime University, Vol.1,1991.

COMMON FIXED POINT THEOREMS ON MENGER SPACES

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1. Introduction

Statistical metric space were introduced by K. Menger in 1942[5]. The author K. Menger looked upon the distance concept as a statistical one rather than a determinate. More precisely, instead of associating a number–the distance $d(p,q)$– with every pair of elements p,q he associated a distribution function $F_{pq}(x)$ which could be interpreted as the probability that the distance from p to q be less than x. In 1960, the space which K. Menger constructed was called the Menger space by B. Schweizer and A. Sklar[8]. And they established topological notions on Menger spaces.

Since metric spaces can be considered as Menger spaces, we may raise a question that a certain kind of fixed point theorems on metric spaces can be generalized for Menger spaces. To answer this question, we need to review contraction mappings on metric spaces and fixed point theorems for such mappings.

The purpose of this paper is to define some generalized contraction mappings on Menger spaces and to obtain various types of common fixed point theorems for families of mappings on Menger spaces.

Many mathematicians have investigated common fixed point theorems for families of mapping on metric spaces ([1],[2],[3],[6],[7],[10]).

In section 2, we define the Menger space. And in section 3, we devote to common fixed point theorems for a family of mapping on the Menger space.

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2. Definitions

Let R denote the reals. A mapping $F : R \to R$ is said to be a *distribution* if it is nondecreasing left-continuous with $\inf F=0$ and $\sup F=1$.

We will denote by L the set of all distribution functions. And H will always denote the specific distribution function:

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

DEFINITION 2.1 ([8],[9]). A statistical metric space (briefly, *SM*-space) is an ordered pair $(S,\mathcal{T})$ where S is an abstract set and $\mathcal{T}$ is a mapping of $S \times S$ into L i.e., $\mathcal{T}$ associates a distribution function $\mathcal{T}(p,q)$ with every pair (p,q) of points in S. We shall denote the distribution function $\mathcal{T}(p,q)$ by F_{pq}, where the symbol $F_{pq}(x)$ will denote the value of F_{pq} for the real argument x. The function F_{pq} are assumed to satisfy the following conditions:

(SM-Ⅰ) $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$,

(SM-Ⅱ) $F_{pq}(0)=0$,

(SM-Ⅲ) $F_{pq}=F_{qp}$,

(SM-Ⅳ) If $F_{pq}(x)=1$ and $F_{qr}(y)=1$, then $F_{pr}(x+y)=1$,

for any p,q,r in S and $x,y \in R$.

Definition 2.1 suggests that $F_{pq}(x)$ may be interpreted as the probability that the distance between p and q is less than x.

DEFINITION 2.2 [9]. A mapping $\Delta:[0,1] = \times [0,1] \longrightarrow [0,1]$ is Δ-norm if it satisfies

(Δ-Ⅰ) $\Delta(a,1)=a,\ \Delta(0,0)=0$

(Δ-Ⅱ) $\Delta(a,b)=\Delta(b,a)$

(Δ-Ⅲ) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a,\ d \geq b$,

(Δ-Ⅳ) $\Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$, for any $a,b,c \in [0,1]$.

Tge following are examples of Δ-norms:

(a) $\Delta_1 : \Delta_1(a,b) = max\,\{a+b-1, 0\}$,

(b) $\Delta_2 : \Delta_2(a,b) = ab$,

(c) $\Delta_3 : \Delta_3(a,b) = min\,\{a,b\}$.

We will denote by $\mathcal{B}$ the set of all Δ-norms.

DEFINITION 2.3 [8]. A Menger space is a triple $(S,\mathcal{T},\Delta)$, where $(S,\mathcal{T})$ is an *SM*-space and $\Delta \in \mathcal{B}$ satisfies the following triangle inequality:

(SM-V) $F_{pr}(x+y) \geq \Delta(F_{pq}(x), F_{pr}(y))$, for all p,q,r in S and for all $x \geq, 0$ $y \geq 0$.

The following theorem establishes a connection between the metric space and the Menger space.

THEOREM 2.4 [9]. Let (S,d) be a metric space. Then the metric d induces a mapping $\mathcal{T}: S \times S \longrightarrow L$, where $\mathcal{T}(p,q)$ $(p,q \in S)$ is defined by $\mathcal{T}(p,q)(x) = F_{pq}(x) = H(x-d(p,q))$, $x \in R$. Further if Δ is $\Delta_3 = \min$, then $(S,\mathcal{T},\Delta)$ is a Menger space and $(S,\mathcal{T},\Delta)$ is complete if the metric space (S,d) is complete.

The space $(S,\mathcal{T},\Delta)$ obtained by a metric space (S,d) is said to be *an induced Menger space*.

3. Common fixed point therems for the family of mappings

We define some generalized contractive mappings on Menger spaces.

DEFINITION 3.1. Let $(S,\mathcal{T})$ be an *SM*-space. A mapping T of S into itself is said to be *contractive type mapping* (briefly *CT-mapping*) if there exists a constant k, $0 < k < 1$ such that for all p,q in S,

$$F_{TpTq}(kx) \geq \min\,\{F_{pq}(x),\ [F_{pTp}(x)+F_{qTq}(x)],\ [F_{pTq}(x)+F_{qTq}(x)]\}$$

for all $x > 0$.

DEFINITION 3.2. Let $(S,\mathcal{T})$ be an *SM*-space. Two mappings T_1 and T_2 of S into itself are said to satisfy the *CT*-mapping condition if there exists a constant k, $0<k<1$ such that for all p,q in S,

$$F_{T_1pT_2q}(kx)\geq min\ \{F_{pq}(x),\ [F_{pT_1p}(x)+F_{qT_2q}(x)],$$
$$[F_{pT_2q}(x)+F_{qT_1q}(x)]\}$$

for all $x>0$.

We reproduce the following fixed point theorems on Menger spaces.

THEOREM 3.1 ([4]). Let $(S,\mathcal{T},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. If T is any *CT*-mapping from S into itself, then T has a unique fixed point p in S.

THEOREM 3.2 ([4]). Let $(S,\mathcal{T},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. If T_1 and T_2 are mappings from S into itself satisfying the *CT*-mapping condition, then T_1 and T_2 have a unique common fixed point p in S.

We obtain the following main results.

THEOREM 3.3. Let $(S,\mathcal{T},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. If $\{T_i\}$ $(i=1,2,\cdots,n)$ is a family of mappings of S into itself such that the composition $T_1T_2\cdots T_n$ is a *CT*- mapping and $T_iT_j=T_jT_i$ $(i,j=1,2,\cdots,n)$, then $\{T_i\}$ has a unique common fixed point p in S.

Proof. Let $T=T_1T_2\cdots T_n$. By the hypothesis, T is a *CT*- mapping. Hence, by the theorem 3.1, T has a unique fixed point p in S. Thus $Tp=p$. Now for $i=1,2,\cdots,n$, $T_ip=T_iTp=TT_ip$. Hence T_ip is a fixed point of T. But since T has a uniqe fixed point p, $T_ip=p$ $(i=1,2,\cdots,n)$. The uniqeness is trivial by the uniqueness of the fixed point of T.

As immediate corollaries, we have the following;

Corollary 3.4 Let $(S,\mathcal{T},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. If $\{T_i\}$ $(i=1,2,\cdots,n)$ is a family of mappings of S into itself such that there exists a constant k, $0<k<1$ such that for all p,q in S, $F_{T_1T_2\cdots T_np\,T_1T_2\cdots T_nq}(kx)\geq F_{pq}(x)$ for all $x>0$ and $T_iT_j=T_jT_i$ $(i,j=1,2,\cdots,n)$. then $\{T_i\}$ has a unique common fixed point p in S.

Corollary 3.5. Let (S,d) be a complete metric space and $\{T_i\}$ $(i=1,2,\cdots,n)$ be a family of mappings of S into itself satisfying the condition: $d(T_1T_2\cdots T_np,T_1T_2\cdots T_nq)\leq k\,d(p,q)$ for all p,q in S and for some k, $0<k<1$ and $T_iT_j=T_jT_i$ $(i,j=1,2,\cdots,n)$. Then $\{T_i\}$ has a unique common fixed point p in S.

Now we obtain a common fixed point theorem for the infinite family of mappings.

Theorem 3.6. Let $(S,\mathcal{T},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x)\geq x$ for each $x\in[0,1]$. If $\{T_i\}$ $(i=1,2,\cdots)$ is a family of mappings of S into itself such that, for each i,j $(i,j=1,2,\cdots)$ with $i\neq j$ two mappings T_i,T_j satisfy the CT-mapping condition, then $\{T_i\}$ has a unique common fixed point p in S,

Proof. For each fixed i,j $(i\neq j)$, by theorem 3.2, T_i and T_j have a unique common fixed point p in S.

Next, we show that all these fixed points coincide. Let p,p' be common fixed points of T_i,T_j $(i\neq j)$ and T_i,T_k $(i\neq k)$ respectively. And suppose that $j\neq k$. $T_jp=p$, $T_ip'=Tp'$ imply that

$F_{pp'}(x)=F_{T_j^np\,T_i^np'}(x)$

$$\geq min\ \{F_{T_j^{n-1}pT_i^{n-1}p'}(x/k),\ [F_{T_j^{n-1}pT_j^n p}(x/k)+F_{T_i^{n-1}p'T_i^n p'}(x/k)],$$
$$[F_{T_j^{n-1}pT_i^n p'}(x/k)+F_{T_i^{n-1}p'T_j^n p}(x/k)]\}$$
$$=min\ \{F_{T_j^{n-1}pT_i^{n-1}p'}(x/k),\ 2,\ 2F_{T_j^{n-1}pT_i^{n-1}p'}(x/k)\}$$
$$=F_{T_j^{n-1}pT_i^{n-1}p'}(x/k) \text{ for all } x>0 \text{ and for some } k,\ 0<k<1.$$

Similarly, $F_{T_j^{n-1}pT_i^{n-1}p'}(x/k) \geq F_{T_j^{n-2}pT_i^{n-2}p'}(x/k^2)$. Hence $F_{pp'}(x) \geq F_{pp'}(x/k^n)$ for all $x>0$ and for $n=1,2,\cdots$. Taking $n \longrightarrow \infty$, $F_{pp'}(x)=1$, for all $x>0$. Hence $p=p'$. This completes the proof.

From theorem 3.6, we have the following two corollaries.

Corollary 3.7. Let $(S,\mathcal{F},\Delta)$ be a complete Menger space and Δ be a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$. If $\{T_i\}$ $(i=1,2,\cdots)$ is a family of mappings of S into itself, such that for each i,j $(i,j=1,2,\cdots)$ with $i \neq j$, two mappings T_i and T_j satisfy the condition: there exists a constant k, $0<k<1$ such that for all p,q in S, $F_{T_ipT_jq}(kx) \geq F_{pq}(x)$, then $\{T_i\}$ has a unique common fixed point p in S.

Corollary 3.8. Let (S,d) be a complete metric space and $\{T_i\}$ $(i=1,2,\cdots)$ be a family of S into itself satisfying the condition: for each i,j $(i,j=1,2,\cdots)$ with $i \neq j$, $d(T_ip,T_jq) \leq kd(p,q)$ for all p,q in S and for some k, $0<k<1$. Then $\{T_i\}$ has a unique common fixed point p in S.

We obtain another common fixed point theorem for the infinite family of mappings as follows.

Theorem 3.9. Let $(S,\mathcal{F},\Delta)$ be a Menger space and Δ be a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,1]$. Let $\{T_i\}$ $(i=1,2,\cdots)$ be a family of CT-mappings of S into itself with a fixed point p_i for each $i=1,2,\cdots$, and T be a mapping of S into itself such that the sequence $\{T_iq\}$ convergers to Tq.

If $p_i \longrightarrow p$, where $p \in S$, then p is a fixed point of T and if T is a CT-mapping, then p is uniquely determined.

Proof. Let ε, λ be positive reals. Since $p_i \longrightarrow p$, there exists an integer N such that $F_{pp_i}(\varepsilon/2) > 1-\lambda/2$ and $F_{TpT_ip}(\varepsilon/2) < 1-\lambda/2$ for all $i > N$. Now, for $i > N$, for some k, $0 < k < 1$,

$$
\begin{aligned}
F_{T_ipT_ip_i}(\varepsilon/2) &\geq \min\{F_{pp_i}(\varepsilon/2k),\ [F_{pT_ip}(\varepsilon/2k)+F_{p_iT_ip_i}(\varepsilon/2k)], \\
&\qquad\qquad [F_{pT_ip_i}(\varepsilon/2k)+F_{p_iT_ip}(\varepsilon/2k)]\} \\
&= \min\{F_{pp_i}(\varepsilon/2k),\ [1+F_{pT_ip}(\varepsilon/2k)], \\
&\qquad\qquad [F_{pp_i}(\varepsilon/2k)+F_{p_iT_ip}(\varepsilon/2k)]\} \\
&= F_{pp_i}(\varepsilon/2k) \\
&> 1-\lambda/2.
\end{aligned}
$$

Hence for $i > N$,

$$
\begin{aligned}
F_{TpT_ip_i}(\varepsilon) &\geq \Delta(F_{TpT_ip}(\varepsilon/2),\ F_{T_ipT_ip_i}(\varepsilon/2)) \\
&\geq \Delta(1-\lambda/2,\ 1-\lambda/2) \\
&\geq 1-\lambda/2 \\
&> 1-\lambda.
\end{aligned}
$$

Thus $T_ip_i \longrightarrow Tp$. Since $T_ip_i = p_i$ and $p_i \longrightarrow p$, by T_2-ness of our space, $Tp = p$. Since T is a CT-mapping, the uniqueness is clear. This completes the proof.

As immediate corollaries, we have the following;

COROLLARY 3.10. Let $(S, \mathcal{T}, \Delta)$ be a Menger space and Δ be a continuous function satifying $\Delta(x,x) \geq x$ for each $x \in [0,1]$. Let $\{T_i\}$ $(i=1,2,\cdots)$ be a family of mappings of S into itself with a fixed point p_i for each $i=1,2,\cdots$, such that for each $i=1,2,\cdots$, there exists a constant k, $0 < k < 1$ such that $F_{T_iqt_ir}(kx) \geq F_{qr}(x)$ for all q,r in S and all $x > 0$ and T be a mapping of S into itself such that the sequence $\{T_iq\}$ converges to Tq. If $p_i \longrightarrow p$, where $p \in S$, then p is a fixed point of T and if T satisfies the condition such for all q,r in S, there exists a constant k, $0 < k < 1$ such that $F_{TqTr}(kx) \geq F_{qr}(x)$ for all x〉0, then p is uniquely determined.

Corollary 3.11. Let (S,d) be a metric space and $\{T_i\}$ $(i=1,2,\cdots)$ be a family of mappings of S into itself satisfying the condition: for each $i=1,2,\cdots$, $d(T_iq,T_ir)\leq k\ d(q,r)$ for all q,r in S and for some k, $0<k<1$. If for all q in S, the sequence $\{T_iq\}$ converges to Tq and if $p_i \longrightarrow p$, where $p\in S$, then p is a fixed point of T and if T satisfies the condition such that there exists a constant k, $0<k<1$ such that $d(Tq,Tr)\leq k\ d(q,r)$ for all q,r in S, then p is uniquely determined.

References

[1] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc, 12(1961), 7-10.

[2] C.E. Hardy and T.D. Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., 16(2),201-206.

[3] K. Iseki, *Fixed point theorems in complete metric space*, Math. Seminar Notes, XXIX, Kobe Univ., 1974, 1-6.

[4] C.W. Kim, *Fixed point theorems on Menger space*, thesis,1983.

[5] K. Menger, *Statistical metric*, Proc. Nat. Acad. Soc. U.S.A., 28(1942), 534-537.

[6] S.B. Nadler, Jr., *Sequence of contractions and fixed points*, Pacific J. Math., 27(3)(1968), 579-585.

[7] S. Reich, *some remarks concerning contraction mapping*, Canad. Math. Bull., 14(1)(1971), 121-124.

[8] B. Schweizer and A. Sklar, *Statistical metric space*, Pacific J.Math., 10 (1960), 313-334.

[9] V.M. Sehgal and A.T. Bharucha-Reid, *Fixed points of contraction mappings on probabilitic metric spaces*, Math. Systems Theory, 6(1972), 97-102.

[10] M.R. Taskovic, *On a family of contractive maps*, Bull. Austral.Math. Soc., 13(1975), 301-308.

Journal of the Research Institute of Basic Sciences,
Korea Maritime University, Vol.1,1991.

THE WEAK ATTOUCH–WETS TOPOLOGY AND THE METRIC ATTOUCH–WETS TOPOLOGY

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Abstract. The purpose of this paper is to find some relations between the weak Attouch-Wets topology and the metric Attouch-Wets topology for the nonempty closed convex subsets of a metrizable locally convex space X. We verify that the former is coaser than the latter. Moreover, we show that X is normable if and only if the two uniformities determining the two topologies for the closed convex subsets of $X \times R$ respectively are equivalent. Our results strengthen and sharpen those of Holá in terms of uniformity itself rather than the topology determined by the uniformity.

I. Introduction

As a successful generalization of the classical Kuratowski convergences of closed convex sets in finite dimensions [8], Attouch-Wets topology [1] in a general normed space X has lately attracted considerable attention. The reason why this topology receives a good deal of attention is that it is stable with respect to duality without reflexivity or even completeness. This Attouch-Wets topology is the topology of uniform convergence of distance functionals on bounded subsets of X, and is well suited for approximation and convex optimization. Its rich developements can be found in the literature[2][4][5].

Recently, Beer [3] defined, in the context of a locally convex space, the

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