Differences between box topology and product topology

Sung Ryong Yoo

Box位相과 Product位相사이의 差異点에 關하여

劉　成　龍

이 論文에서 位相空間들의 cartesian product에 對해서 box product 位相을 導入하였다. 有限個의 product에 關해서는, box 位相과 product 位相은 相等하며 無限個의 product에 關해서는, box 位相이 product 位相을 包含하는 空間이다.

box 位相에서는 數列의 收斂性은 保存되지 않으며 ΠX_α가 連結空間면 各各의 X_α가 連結空間이나 그 逆은 成立되지 않는 例를 들어 밝혔다.

또한 ΠX_α가 compact일 必要充分條件은 各各의 X_α가 compact이고 X_α의 有限個를 除外한 나머지가 모두 singleton임을 證明해 보았다.

Abstract

In this paper, we introduce that a base for the box topology for the cartesian product set $\Pi\{X_\alpha;\alpha\in A\}$ is the family of all sets $\Pi\{U_\alpha;\alpha\in A\}$ where U_α is open in X_α for each α in A.

From the definition, when A is finite, the box topology is identical with the product topology. When A is infinite, situations are different. I will work mainly with the case when A is infinite.

It is shown that the convergence of sequences is not preserved by box product; If ΠX_α is connected then each X_α is connected. The converse of this need not be true. And ΠX_α is compact iff each X_α is compact and all but finite number of coordinate space are singleton.

Definition 1. Let (X_α, T_α) be a topological spaces for each $\alpha\in A$.

A base for the box topology on $\Pi\{X_\alpha;\alpha\in A\}$ is the family of all sets of the form $\Pi\{U_\alpha;\alpha\in A\}$ where $U_\alpha\in T_\alpha$ for each $\alpha\in A$.

It can be easily seen that the family of such sets actually form a base for a topology.

When A is finite, box topology and product topology on ΠX_α coincide from the definitions. In general, box topology is finer then the product topology. When A is infinite, box topology is quite different from the product topology.

In the following P and T denote the product topology and the box topology on $\Pi\{X_\alpha;\alpha\in A\}$ where each (X_α, T_α) is a topological space, respectively. I also adopt the convention that the space ΠX_α means the space ΠX_α with box topology other wise specified.

There are some properties common to both topologies as shown in theorem 1,2 and lemma 1.

Theorem 1. Each projection from a box product space to its coordinate spaces is continuous and open.

Proof; Each projection $P_\alpha:(\Pi X_\alpha, P)\to X_\alpha$ is continuous.

Since T is finer than P, it is clear that $P_\alpha:(\Pi X_\alpha, T)\to X_\alpha$ is continuous for each $\alpha\in A$.

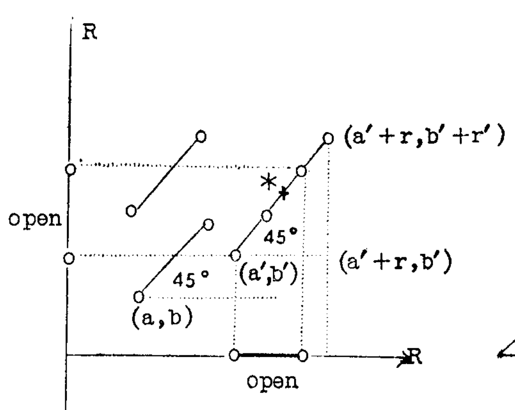
And then, to show that each projection is open, it suffices to show that images of basic open sets are open.

Let $V=\Pi\cup_\alpha$ be a basic open set. Then $P_\alpha(V)=\cup_\alpha$ which shows $P_\alpha(V)$ is open in X_α.

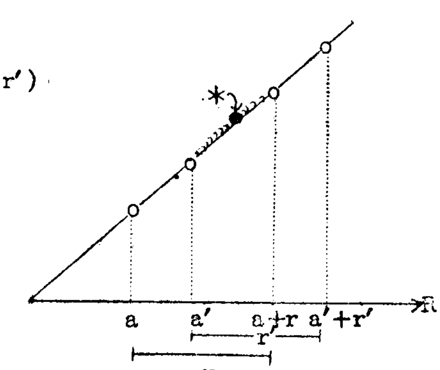
Of course, box topology is not the smallest topology that makes each projection continuous unless the number of coordinate spaces is finite. And as the following example shows it need not be the finest topology that makes each projection open.

Example 1. There is finer topology than box topology, which makes each proection open.

In $R^2=R\times R$(product topology=box topology)$\$=\{(a+t, b+t); 0<t<r, a,b\in R, r>o\}$. Let us denote $\{(a+t,b+t);o<t<r\}$ by Vabr, if $*$ belongs to Vabr$\cap$Va′b′r′ then $*$ belongs to Vcds, where $\max(a,a')=c$ $\max(b,b')=d$, $s=\min(a+r-c, a'+r'-c)$, as shown below. With the aid of example 1, finest topology(makes each projection



(Fig-1)



(Fig-2)

open) includes properly box topology.

Lemma 1. If $A_\alpha \subset X_\alpha$, then $\overline{\Pi A_\alpha} = \Pi \overline{A}_\alpha$

Proof; If x belongs to $\overline{\Pi A_\alpha}$ then for all $V(x)$, $V(x) \cap \Pi A_\alpha \neq \phi$ and then $X_\alpha = P_\alpha(x)$, ${}^\forall U_\alpha \ni x_\alpha$, $P^{-1}[U_\alpha(x_\alpha)]$ is nbd. of x. $P_\alpha^{-1}[U_\alpha(x_\alpha)] \cap \Pi A_\alpha \neq \phi$. And so $U_\alpha(x_\alpha) \cap A_\alpha \neq \phi$ and then x belongs to $\overline{A}_\alpha$.

Hence $x = (x_\alpha, \cdots)$ belongs to $\Pi \overline{A}_\alpha$

Conversely, $x \in \Pi \overline{A}_\alpha \rightarrow x_\alpha \in \overline{A}_\alpha \rightarrow {}^\forall U_\alpha(X_\alpha) \cap A_\alpha \neq \phi$. On the other hand, U; arbitrary basic nbd of x, $U = \Pi \cup_\alpha (\cup_\alpha$; nbd of $x_\alpha)$, $\cup \cap \Pi A_\alpha \neq \phi$. Since $\cup_\alpha \cap A_\alpha \neq \phi$. Hence $x \in \overline{\Pi A_\alpha}$.

Theorem 2. If $A_\alpha \subset X_\alpha$ for each $\alpha \in A$, then ΠA_α with box product of A_α(subspace of X_α) is identical with the subspace ΠA_α of ΠX_α.

Proof; Let$(\Pi A_\alpha, T_1)$ be the box product space of A_α and$(\Pi A_\alpha, T_2)$be the subspace of box space ΠX_α. $U \in T_1$ iff $U = \Pi V_\alpha; V_\alpha$ open in A_α, $V_\alpha = U_\alpha \cap A_\alpha$, $U_\alpha \in T_\alpha$

$$U = \Pi(U_\alpha \cap A_\alpha)$$
$$= \Pi(U_\alpha) \cap (\Pi A_\alpha)$$

$\Leftrightarrow U = (\Pi U_\alpha) \cap (\Pi A_\alpha)$ is open in ΠA_α, since $\Pi \cup_\alpha \in \Pi T_\alpha \Leftrightarrow U \in T_2$

Corollary. Let $A_\alpha \subset X_\alpha$ for each α. ΠA_α is dense in ΠX_α iff A_α is dense in X_α for each $\alpha \in A$.

Remark. Let$X$ be box product, and $x^o = \{x_\alpha^0\}$ a given point.

For each index, the set, $S(x^o, \beta) = X_\beta \times \Pi\{X_\alpha^o \mid \alpha \neq \beta\} \subset \Pi X_\alpha$, is called the slice in ΠX_α through x^o parallel to X_β.

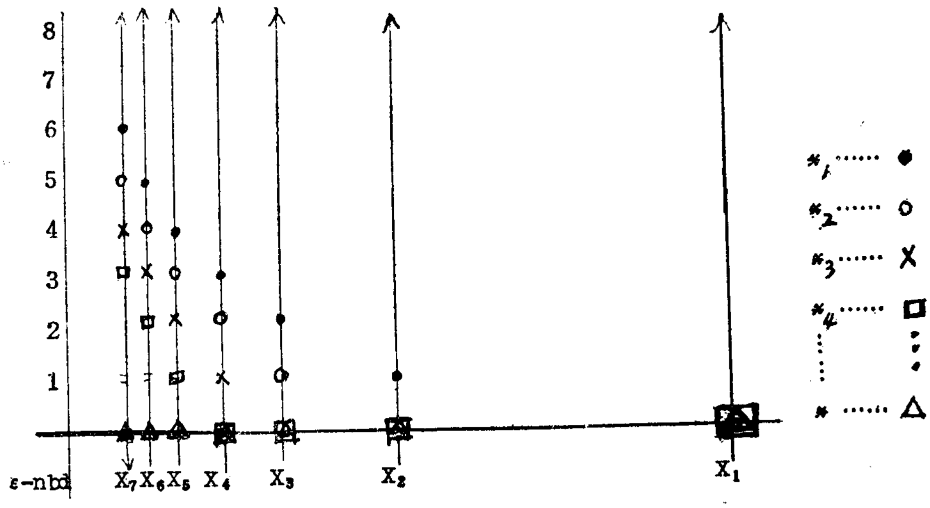
From now on, I derive some of basic properties for box topology.

The following example shows that the box product does not preserve convergence of sequence.

Example 2. Let X be the space $\Pi\{X_n \mid n \in Z^+\}$ with box toplogy. Let a sequence$\{x_n \mid n \in Z^+\}$ in ΠX_n be given. For each $n \in Z^+$, let the sequence$\{x_n; n \in Z^+\}$ where $x_n = \{x_{1n}, x_{2n}, \cdots\}$ converge to a point x_n in X_n. But the sequence $\{x_n; n \in Z^+\}$need not converge to x. Observe, as illustrated below, that the projection $<P_\alpha(x_n)$ of $<x_n>$ into coordinate space converge to zero.

$$x_1 = (0, 1, 2, 3, 4, 5, \cdots\cdots)$$
$$x_2 = (0, 0, 1, 2, 3, 4, 5, \cdots\cdots)$$
$$x_3 = (0, 0, 0, 1, 2, 3, 4, 5, 6, \cdots\cdots)$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$x_n = 0, 0, 0, \cdots\cdots, \; 0, 1, 2, 3, 4, 5, \cdots\cdots)$$
$$\downarrow\;\downarrow\;\downarrow \qquad \downarrow\;\downarrow \;\cdots\; \downarrow \cdots$$
$$x = (0, 0, 0, \cdots\cdots, \; 0, 0, \cdots\cdots 0, \cdots\cdots)$$

Each $P_\alpha(x_n)$, $n \in Z^+$, converges to zero. But $\{x_n\}$ does not converge to $x_n = (0, \cdots\cdots, 0, \cdots\cdots)$. Because we choose a ε-nbd of x as $\prod_{n=1}^{\infty}$ $(-1, 1)$, and $\prod_{n=1}^{\infty}$ $(-1, 1)$ is base on



(Fig-3)

box topology. In fact x does not belong to $\prod_{n=1}^{\infty}(-1,1)$

But if we give the usual product topology to $\prod_{n=1}^{\infty} X_n$, then $<x_n>$ converges to x.

Since I can choose an nbd of x in product space as $\prod_{n=1}^{n}(-1,1)\times\Pi\{X_\alpha;X_\alpha\neq(-1,1)\}$, then x belongs to $(-1,1)\times(-1,1)\times X_{n+1}\times X_{n+2}\times\cdots\cdots$

The product topology has a nice property such as: arbitrary product of compact spaces is compact(Tychonoff theorem). When box topology is given, many properties are preserved only when the number of coordinate spaces is finite(and when it is the case, box topology is identical with product topology).

Lemma 2. ΠX_α is 2°-countable iff each X_α is 2°-countable and all but finite number of coordinate spaces are singleton.

Theorem 3. ΠX_α is compact iff each X_α is compact and all but finite number of coordinate spaces are singleton.

Proof; If ΠX_α is compact, then each X_α is compact. Since each projection $P_\alpha;\Pi X_\alpha\to X_\alpha$, is continuous surjection and we already knew that the continuous in image of a compact set is compact. Hence X_α is compact. And now, we show that all but finite number of coordinate spaces are singleton, assume that there are infinite number of coordinate spaces with cardinality not less then 2, that is, $A=\{\alpha;|X_\alpha|\geq 2\}$ is infinite.

Let x_n, x'_n be two distinct points in X_n for each n, U_n be $X_n-\{x_n\}$ and $V_n=X_n-\{x'_n\}$.

Then U_n and V_n are open. $A=\{<\prod_{i=1}^{n}W_i>;W_i=U_i$ or $V_i\}$ is an infinite open cover which has no finite subcover, because if A have a finite subcover, $<\Pi W_i^1>$, $<\Pi W_i^2>$,

$\cdots, <\Pi W^k_i>$, and let $z=<z_i>$ where $z_i=$

$$\begin{cases} x'_i & \text{if} \quad W_i=U_i \\ x_i & \text{if} \quad W_i=V_i \end{cases}$$

Then z does not belong to $\bigcup_{j=1}^{k} <\Pi W^j_i>$.

Converse is clear from the tychonoff theorem.

Lemma 3. ΠX_α is separable iff each X_α is(1) separable and (2) all but finite numebr of coordinate spaces are singleton.

Proof; If ΠX_α is separable then each X_α is (1) separable, since $P_\alpha;\Pi X_\alpha \to X_\alpha$, is continuous, and continuous image of a separable space is separable.

Hence each X_α is separable. And if ΠX_α is separable, then each X_α is (2) all but finite number of coordinate spaces are singleton, since for each $\beta \in \mathfrak{B}$ where $\mathfrak{B}$ is $\{\alpha \in A; X_\alpha \geq 2\}$, there exist U_β and V_β such that they are nonempty, disjoint and open sets in X_β.

Let D be a countable dense subset of ΠX_α, and for all $\mathfrak{B}' \subset \mathfrak{B}$, let W_β be $\langle \Pi_{\beta \in \mathfrak{B}} W_\beta \rangle$ i.e.,

$$W_\beta = \begin{cases} U_\beta \text{ if } \beta \text{ belong to } \mathfrak{B}' \\ V_\beta \text{ if } \beta \text{ do not belong to } \mathfrak{B}' \end{cases}$$

For example, $\Pi X_\alpha = X_1 \times X_2 \times X_3 \times X_4 \times \cdots\cdots \times X_7 \times \cdots\cdots$ Where $X_2, X_4, \cdots\cdots, X_7, \cdots$ are singleton.

$A=\{1,2,3,4,5,6,7,8,\cdots\cdots\}$

$\mathfrak{B}=\{1,3,5,6,8,9,\cdots\cdots\cdots\cdots\}$

$\mathfrak{B}'=\{1,6,9,\cdots\cdots\cdots\cdots\cdots\cdots\}$

$W_{\beta'}=\langle V \times U \times V \times U \times V \times U. \times \cdots\cdots \rangle$

$= V_1 \times X_2 \times U_3 \times X_4 \times V_5 \times U_6 \times X_7 \times \cdots\cdots$

$\subset \prod_{n=1}^{\infty} X_n$

And so, it is sufficient to showt hat $\{W_{\beta'};\mathfrak{B}' \subset \mathfrak{B}\}$ is nonempty, disjoint and open.

By assumption, $\{W_{\beta'};\mathfrak{B}' \subset \mathfrak{B}\}$ is nonempty. Only to show disjoint, if $\mathfrak{B}'$ is not $\mathfrak{B}''$, then $W\mathfrak{B}' \neq W\mathfrak{B}''$. Since if $\mathfrak{B}'$ is not $\mathfrak{B}''$ then there is β_0 such that β_0 belongs to $(\mathfrak{B}'-\mathfrak{B}'')$.

And $P_{\beta_0}(W_{\beta'})=U_{\beta_0}$, $P_{\beta_0}(W_{\beta''})=V_{\beta_0}$, $U_{\beta_0} \cap V_{\beta_0}=\phi$ in X_β.

And hence $W_{\beta'} \cap W_{\beta''} \subset \langle U_{\beta_0} \rangle \cap \langle V_{\beta_0} \rangle = \langle U_{\beta_0} \cap V_{\beta_0} \rangle = \phi$.

And for all $\mathfrak{B}' \subset \mathfrak{B}$, We pick up a point $d_{\beta'}$ which exists in $D \cap W_{\beta'}(W_{\beta'}$; open in $\Pi X_\alpha)$

then $P(\mathfrak{B}) \to D$ and so,

$\mathfrak{B}' \to d_{\beta'}$ is an injection.

Hence $Z(P(\mathfrak{B}) \leq Z(D) \leq Z_0$

If $Z(\mathfrak{B})=Z_0$ then $Z(P(\mathfrak{B}))=2^{Z_0}>Z_0$. This is a contradiction.

Thus, $Z(\mathfrak{B})<Z$.

Converse is clear; since finite box product is identical with product space $\Pi\{X_\alpha;\alpha\in A\}$ is separable when A is finite.

Lemma 4. If ΠX_α is connected, then each X_α is connected.

But we give an interesting example that shows the converse of Lemma 4. need not be true.

Example 3. Let Y be the cartesion product of the real numbers an infinite of times; that is, $Y=R^A$, where R is the set of real numbers and A is an infinite set. With the box topology Y does not satisfy the first countability axiom, and the component of Y containing y is $\{x\varepsilon Y;\ \{a;\ x_a\neq y_a\}$ is finite$\}$

proof; Let x and y be points of Y whose coordinates differ for an infinite set a_0, $a_1,\cdots\cdots$, $a_p\cdots\cdots$ of members of A. Let Z be the set of all z in Y such that for some k, $p|z(a_p)-x(a_p)|/|x(a_p)-y(a_p)|<k$ for all p, that is, $Z=\bigcup_{k>0}\{z\varepsilon R^A\ ;\ p|z_{a_p}-x_{a_p}|<k|x_{a_p}-y_{a_p}|=\bigcup_k<\Pi x_{a_p}-\frac{k}{p}|x_{a_p}-y_{a_p}|,\ x_{a_p}+\frac{p}{k}|x_{a_p}-y_{a_p}|>$,

then clearly, Z is open.

And now, to show Z is closed, let w be a element in $\bar{Z}$, then there exsists a element z which belongs to $\langle\Pi(w_{a_p}-\frac{1}{p}|x_{a_p}-y_{a_p}|,\ w_{a_p}+\frac{1}{p}|x_{a_p}-y_{a_p}|\rangle$ and Z at the same time.

(1) For all p, $w_{a_p}-\frac{1}{p}|x_{a_p}-y_{a_p}|<z_{a_p}<w_{a_p}+\frac{1}{p}|x_{a_p}-y_{a_p}|$, hence $z_{a_p}-\frac{1}{p}|x_{a_p}-y_{a_p}|<w_{a_p}<z_{a_p}+\frac{1}{p}|x_{a_p}-y_{a_p}|$.

(2) Since z belongs to Z, for all p, $x_{a_p}-\frac{k}{p}|x_{a_p}-y_{a_p}|<z_{a_p}<x_{a_p}+\frac{k}{p}|x_{a_p}-y_{a_p}|$

According to (1) and (2), $x_{a_p}-\frac{k+1}{p}|x_{a_p}-y_{a_p}|<w_{a_p}<x_{a_p}+\frac{k+1}{p}|x_{a_p}-y_{a_p}|$, thus w belongs to Z.

And so Z is open and closed, $x\varepsilon Z$ and $y\notin Z$.

References

1. Bourbaki, N. (1940), Topologie Generale. Chapitrel et II. Actualites Sei. Ind., 858.
2. Dugundji, J., Topology (Allyn, Bacon; Boston, Mass., 1966)
3. J.L. Kelley, General topology, Van Nostrand, N.J., 1955, 107.
4. C.J. Knight, Box topologies, Quart. J. Math. Oxford(2) 15(1964) 41—54.
5. M.E. Rudin, The box topologies, Proc. Univ. of Houston, Point set top. Conf., 1971—199.
6. H. Tamano. Paracompactness of strong products, Notices, AMS. 15(1968)345.
7. W.J. Thron, Topological structures. Holt, Rinehart and Winston Inc., 1966.