Depth Two Subfactors and Hopf * – Algebras

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Abstract

We present some well-known results concerning the irreducible depth two subfactors and related finite dimensional Hopf * – algebras. Recent study of Hopf * – algebras coming from the irreducible depth 2 subfactors $P^H \subset P \rtimes K$ is described.

1. Introduction

Over the past ten years the theory of subfactors has been among the most active topics in Operator Algebras, and it has been linked with other fields of Mathematics and Mathematical Physics.

The theory of Operator Algebras is based on the natural generalization of the complex matrix algebra $M_n(C)$. It was initiated by Murray and von Neumann in the 1930s. An Operator Algebra is a * – algebra of bounded linear operators on a Hilbert space over the complex numbers. Here * refers to the adjoint of an operator, which is a generalization of an adjoint matrix. Consequently, our theory deals with infinite dimensional and non – commutative objects. When an operator algebra contains an identity operator and is closed relative to the weak operator topology, it is called a von Neumann algebra. On the other hand, an operator algebra closed under norm topology is called a C^* – algebra.

A von Neumann algebra is called a factor if its center is scalar multiples of the identity. The full matrix algebra as well as the set of all bounded operators on a Hilbert space are obviously factors. When M denotes a von Neumann algebra acting on a Hilbert space H, its commutants M′ consists of all elements x of B(H) satisfying xy=yx for all $y \in M$. Here B(H) denotes the set of all bounded operators on a Hilbert space H. In 1929, von Neumann obtained that a von Neumann algebra is characterized by the fact that $M=M''$. Unlike C^* – algebras, a von Neumann algebra contains an abundance of projections. By analyzing the structure of the projections, factors can be classified into type I_n, I_∞, II_1, II_∞ and III. Among those factors, a type II_1 factor has continuous dimensions and always admits a unique normalized trace which provides an alternative definition of a type II_1 factor.

Although several remarkable results of classifying II_1 factors have been made, definite results still remain in the dark. It is known recently that many classification problems in the classical Operator Algebras are reduced to classification problems of subfactors. V. Jones′ index theory serves as an important tool for such a problem. One line of investigation in index theory is to understand the structure of a factor von Neumann algebra M, viewed as a finite projective module over its subfactor N. This problem is far from solved. However, some important contributions in this direction have been already made. In particular, it has been observed by A. Ocneanu that subfactors of depth two can be represented as crossed products by outer actions of finite dimensional Hopf *－algebras. Since then, many applications of Hopf *－algebras to Operator Algebras have been found.

In this notes, we review some results concerning the depth two subfactors as well as the structure of the related finite dimensional Hopf *－algebras. Throughout this paper, we denote by C the complex numbers.

2. Hopf *－algebras and their actions

A finite dimensional Hopf *－algebra H is a finite dimensional C^*－algebra equipped with linear maps

(i) comultiplication $\Delta : H \to H \otimes H$

(ii) counit $\varepsilon : H \to C$

(iii) antipode $S : H \to H$

satisfying the relations ; $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, $(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta$, and $m(S \otimes id)\Delta = \varepsilon = m(id \otimes S)\Delta$, where id denotes the identity mapping of H and $m : H \otimes H \to H$ denotes the multiplication. Here, both Δ and ε are C^*－algebra homomorphisms and S is a *－preserving antimultiplicative involution.

Dual algebra H° is the vector space of linear functionals on a finite dimensional Hopf algebra H, which turns out again a finite dimensional Hopf *－algebra with the following structure :

$(\phi \cdot \varphi)(h) = (\phi \otimes \varphi)\,\Delta(h)$, $\phi^*(h) = \overline{\phi(S(h^*))}$, $(\Delta\phi)(h \otimes g) = \phi(hg)$, $\varepsilon^\circ(\phi) = \phi(I)$, and $S^\circ(\phi)(h) = \phi(S(h))$, for $\phi, \varphi \in H^\circ$.

A Hopf algebra H is commutative in case $mT = m$, and cocommutative in case $T\Delta = \Delta$, where $T : H \otimes H \to H \otimes H$ is the twisted map given by $T(a \otimes b) = b \otimes a$.

Examples. (1) The simplest example of a cocommutative Hopf algebra is the complex group algebra $C[G]$ of a finite group G , equipped with $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $S(g) = g^{-1}$.

(2) The simpleast example of commutative Hopf algebra is the function algebra $C(G)$ with minimal projections $\{p_g \mid g \in G\}$. This is dual to the group algebra $C[G]$, with comultiplication $\Delta(p_g) = \sum_{h \in G} p_h \otimes p_{h^{-1}g}$, counit $\varepsilon(p_g) = \delta_{eg}$, and antipode $S(p_g) = p_{g^{-1}}$. Here e denotes the netural element of G.

(3) $C(G) \otimes C[G]$ is obviously a non－commutative and non－cocommutative Hopf algebra when G is not abelian.

There are but few general methods of constructing nontrivial non－commutative and non－cocommutative finite dimensional Hopf algebras. One of the most important is due to recent works by S. Majid [6]. He constructed Hopf algebras as bicrossproducts, thus producing examples which are neither commutative nor cocommutative.

An action of finite dimensional Hopf algebra H on a unital *－algebra A is a bilinear map $\cdot : H \times A \to A$ satisfying the following conditions ; $h \cdot I = \varepsilon(h)$, $I \cdot x = x$, $hg \cdot x = h \cdot (g \cdot x)$, $h \cdot xy = \sum_i (h_i^L \cdot x)(h_i^R \cdot y)$, $(h \cdot x)^* = S(h^*) \cdot x^*$, for all $h, g \in H$, and $x, y \in A$. Here we denoted by $\Delta(h)$ by $\sum_i h_i^L \otimes h_i^R$. This is exactly a group action when Hopf algebra H is replaced by a finite group G.

Example. Cocommutatative Hopf algebra H＝C[G] acts on A via a bilinear map $\alpha : G \to Aut(A)$ given by $(\sum_{g \in G} \mu_g g) \cdot x = \sum_{g \in G} \mu_g \alpha_g(x)$.

The crossed product $A \times H$ is a unital *－algebra $A \otimes H$(coincides with a vector space), with multiplication and * - operation defined by

$(x\otimes h)(y\otimes g)=\sum_i x(h_i^L \cdot y)\otimes h_i^R g,$

$(h \cdot x)^*=S(h^*)\cdot x^*.$

The fixed point algebra is defined as $A^H=\{x\in A \mid h\cdot x=\varepsilon(h)x,\ \forall h\in H\}$. If all elements of A are fixed, then the action is trivial. As the group action case, the action is called outer when$(A^H)'\cap A=CI$. It is known that any finite dimensional Hopf * - algebra acts outerly on the hyperfinite type II_1 factor.

3. Depth two subfactors

Let M be a type II_1 factor with a unique normalized trace τ. When M is acting on a Hilbert space H, one can regard H as an M - module. Therefore we can define the Murray von Neumann coupling constant dim ${}_MH$ for M on H, by using the trace τ. Given a subfactor N of M with the same identity, V. Jones defined the index [M : N] based on a coupling constant $\dim_N L^2(M)$, where the Hilbert space $L^2(M)$ is the GNS completion of M with respect to the inner product $\langle a,b\rangle=\tau(ab^*)$([2]).

When e_N denotes the orthogonal projection from $L^2(M)$ onto the closed subspace of L^2(N), V. Jones have constructed the(basic construction) von Neumann algebra $M_1=\{M, e_N\}''$, generated by M and e_N , acting on $L^2(M)$. If N and M are factors then $[M : N]<\infty$ if and only if M_1 is a type II_1 factor and $[M_1 : M]=[M : N]$. Also we have $[M : N]=\tau(e_N)^{-1}$([2]).

Assume that $[M : N]<\infty$. By letting M_1 act on its own standard Hilbert space $L^2(M_1)$ we can construct the next basic construction $M_2=\{M_1,e_M\}''$ of $M\subset M_1$, with the orthogonal projection e_M onto the close subspace of $L^2(M)$. We see that $[M_2 : M_1]=[M_1 : N]$ and M_2 is a type II_1 factor. Iterating this procedure, we obtain a tower of type II_1 factors

$$N\subset M\subset M_1\subset M_2\subset M_3\subset\cdots\cdots,$$

and consequently a derived tower of relative commutants

$$CI=N'\cap N\subset N'\cap M\subset N'\cap M_1\subset N'\cap M_2\subset\cdots\cdots.$$

When the index [M : N] is finite, the relative commutants $N'\cap M_n$ turn out to be finite dimensional. Moreover, each algebra $N'\cap M_n$ contains the basic contruction of $N'\cap M_{n-2}\subset N'\cap M_{n-1}$ $(n\geq 2)$ as a subalgebra , where $N=M_{-1}$, $M=M_0$. Due to finite dimensioal property, the derived tower can be described easily with help of Bratteli diagram. Moreover, the mirror image of the previous inclusionis always part of the next inclusion in the derived tower. Thus we may throw away all the reflected stuff and look at the remaining part. The remaining part of the diagram is called a principal graph for $N\subset M$, which plays an important invariant for the studying the structure of the original pair. The depth of $N\subset M$ is defined by the length of the principal graph from the distinguished vertex (associated with the commutant $CI=N'\cap N$). If the principal graph is finite, then the pair $N\subset M$ is said to have finite depth. Otherwise, $N\subset M$ is said to have infinite depth.

When a finite group G acts outerly on a type II_1 factor M, Hopf algebra techniques can be useful in the investigation of the derived tower for $N=M^G\subset M$. Indeed, if the group G is abelian, it is well - known that $M_1=M\times G$, and $M_2=M_1\times\hat{G}$, with the dual action of the dual group $\hat{G}$ of the abelian group G. Moreover, we have $N'\cap M_1=C[G]$, $M'\cap M_2=C[\hat{G}]=C(G)$, and $N'\cap M_2=M_{|G|}(G)$, the matrix algebra([2]). That is, $N\subset M$ has depth two. This description remains valid even if we no longer require that G be abelian, when we replace $\hat{G}$ with C(G), Hopf * - algebra dual to C[G]. As indicated by A. Ocneanu, and proved by W. Szymanski([7], see [5] for infinite factors), this model characterizes the tower of factors as follows.

Theorem 1([7]) Let $N \subset M$ be type II_1 factors with $[M : N] < \infty$ and $N' \cap M = CI$. If $N' \cap M_1$ is simple, then $M' \cap M_2$ has a natural Hopf * – algebra structure and acts outerly on M_1 such that M_2 is isomorphic to the crossed products $M_1 \rtimes (M' \cap M_2)$.

Note that the algebra $N' \cap M_1$ is dual to $M' \cap M_2$, and it follows from the duality between crossed products and the fixed point algebras that the algebra $N' \cap M_1$ acts outerly on M so that $N = M^{N' \cap M_1}$.

The structure of the involved Hopf * – algebras are trivial ones if we consider the pairs related to crossed products or fixed point algebras. When $N = M^G \subset M$, the Hopf algebra $N' \cap M_1$ is the group algebra $C[G]$. When $N \subset N \rtimes G = M$, the Hopf algebra $N' \cap M_1$ is the function algebra $C(G)$([2]). The nontrivial examples of non – commutative and non – cocommutative Hopf * – algebras for $N' \cap M_1$ can be obtained by considering composition of two depth 2 subfactors. The study of such subfactors has been recently initiated by D. Bisch and U. Haagerup [1].

Let G be a group of outer automorphisms of a type II_1 factor P. When finite subgroups H and K of G act outerly on P, $N = P^H \subset P \rtimes K = M$ are type II_1 factors with $[M : N] = |H||K|$. The derived tower of the pair was analyzed in [3, 4]. In particular, the depth two case has been clarified in that situation, as follows.

Theorem 2([4]) Let H and K be finite subgroups of the group G . Then $N = P^H \subset P \rtimes K = M$ has depth two if and only if H and K satisfy(i) $H \cap K = \{e\}$ (ii) HK is a group in G, where e denotes the neutral element of G.

Theorem 3([4]) Let $N = P^H \subset P \rtimes K = M$ have depth two. Then the Hopf algebra $N' \cap M_1$ has the structure of a twisted bicrossproduct.

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