CS–semistratifiable 空間에 關하여(其 II)

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2nd note on cs-semistratifiable space

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Abstract

In this note we intrduce the concept of cs-semistratifiable spaces, and we make mention of some properties for cs-semistratifiable space. We also show that the countable product of cs-semistratifiable spaces is cs-semistratifiable.

We relate the classes of cs-semistratifiable spaces to the likely characterizations with the more familiar classes of spaces. Finally it is verified that a first countable cs-semistratifiable is stratifiable and that a compact cs-semistratifiable space is complete.

要 約

本論文에서는 cs-semistratifiable 空間의 槪念을 導入하여 그 空間에 關한 몇개의 性質, 可算個積空間이 cs-semistratifiable 空間임을 보이고 cs-semistratifiable 空間을 다른 類似한 空間과의 特性에 關係를 맺은 結果, 第1可算個의 cs-semistratifiable 空間이 stratifiable 空間이 됨을 밝히고 compact cs-semistratifiable 空間이 complete가 되는 事實을 證明했다.

1. Definition

A cs-semistratification for a topological space X is defined as a mapping g from $N\times X$ to the topology of X which satisfies the following conditions;

1° $x\in g_n(x)$

2° $g_{n+1}(x)\subset g_n(x)$

3° If a sequence $\{x_n\}$ converges to a unique point x, then $\bigcap_{i=1}^{\infty} g_i(\langle x;\ x_n\rangle)=\langle x;x_n\rangle$ Here we used the notation $\langle x;\ x_n\rangle=\{x\}\cup\langle x_n\rangle$. Where $\langle x_n\rangle$ denotes the ranges of the sequence $\{x_n\}$.

Also we introduce $g_n(S)=\cup\{g_n(x)\mid x\in S\}$ for any subset S of X. From now on, all space are assumed to be T_2—space.

2. Some properties of cs-semistratifiable spaces.

As we would expect, we make mention of some properties for cs-semistratifiable spaces.

〔Lemma 2.1〕 Every subspace of a cs-semistratifiable space is cs-semistratifiable.

Proof: Trivial by definition.

〔Lemma2.2〕 If X is cs-semistratifiable space then there is a semistratifiable function g with a following condition;

Given a convergent sequence $x_n\to x$ and a open subset U containing x, there exists a $k\in N$ such that $x\notin \bigcup_{x\in X-U} g_k(x)$ and $\{n\in N;\ x_n\in \bigcup_{x\in X-U} g_k(x)\}$ is finite. In this case, g is called a cs-semistratifiable function.

Proof: Let a cs-semistratification f be given. For each $n\in N$ and $x\in X$, define $g_n(x)=X-f_n(X-cl\{x\})$, Creede[4] proved g is a semistratifiable function. To show g satisties the above condition, consider the following $\bigcup_{x\in V} g_k(x)=\bigcap_{x\in V}\{X-f_k(X-cl\{x\})=X-\bigcap_{x\in V} f_k(X-cl\{x\})$ which is contained in $X-f_k(V)$.

If $\{x_n\}$ is eventually in $f_k(V)$, $\{n\in N;\ x_n\in \bigcup_{x\in V} g_k(x)\}$ is finite. For the converse, Let $f_n(U)=X-\bigcup_{x\in X-U} g_n(x)$. Hence $f_n(U)$ is cs-semistratifiable.

〔Theorem2.3〕 the countable product of cs-semistratifiable spaces is cs-semistratiable.

Proof: For each $i\in N$, let X_i be a space with cs-semistratifiale function g_i.

Let $X=\prod_{i=1}^{\infty} X_i$ and let p_i be the projection of X onto X_i. For each $i,\ j\in N$ and $x\in X$, let $hi(j,\ x)=g_i\{j,\ p_i(x)\}$ if $i\leqq j$ and $hi(j,x)=X_i$ if $i>j$. Now that $g(j,\ x)=\prod_{i=1}^{\infty} h_i(j,\ x)$ for each j and x.

By definition 1 and Lemma 2.2, It is clear to verify g is a cs-semistratifiable function for space X.

To show g satisfies the condition of Lemma 2.2, let $\{x_n\}$ be a sequence converging to z and let $z\in U\in\mathscr{T}$. Take a basic open neighborhood V of z. $V=\prod_{i\in F} V_i\times\prod_{i\in N-F} X_i\subset U$, where F is a finite subset of N. For each i, $\{p_i(x_n);\ n\in N\}$ is a sequence converging to $p_i(z)$, and $p_i(V)$ is open in X_i and contains $p_i(z)$, There is a k_i such that $\{n\in N\mid p_i(x_n)\in\cup\{g_i(k_i,s)\mid s\in X_i-p_i(V)\}$ is finite for each $i\in F$. Let $k=\max\{k_i\mid i\in F\}$. But $x_n\in$

$\bigcup_{x \in X-V} g_k(x)$ iff there is a $x \in X-V$ such that $x_n \in g_k(x)$ iff there is a $x \in X$ such that $p_i(x) \in X_i - p_i(V)$ for some $i \in F$ and $x_n \in g_k(x)$ If and only if there exists a $x \in X$ such that $p_i(x) \in X_i - p_i(V)$ for some $i \in F$ and $p_i(x_n) \in g_i\{k, p_i(x)\}$ This implies $p_i(x_n) \in \bigcup \{g_i(k, s) | s \subset X_i - p_i(V)\}$. Thus $\{n \subset N | x_n \in \bigcup_{x \in X-V} g_k(x)\}$ is finite. This insures that $\{n \in N | x_n \in \bigcup_{x \in X-V} g_k(x)\}$ is finite since $V \subset U$.

[Corollary 2.4] X is cs-semistratifiable if X is paracompact and locally cs-semistratifiable.

Proof: For each $x \in X$, there exists an open neighborhood $W(x)$ of x such that $W(x)$ is cs-semistratifiable. By paracompactness of X, there is a locally finite closed refinement $\{C_\alpha;\ \alpha \in \mathscr{A}\}$ of $\{W(x) | x \in X\}$. Then each C_α is cs-semistratifiable by lemma 2.1

3. Some characterizations

Finally we relate the class of cs-semistratifiable spaces to the nice characterizations with the more familiar classes of spaces.

Let g be a map from $N \times \mathscr{T}$ to the family of all closed subsets of a spaces $(X, \mathscr{T})$. consider the following conditions on g;

S_1 For each $U \in \mathscr{T}$, cl $g_n(U) \subset U$

S_2 If U, $V \in \mathscr{T}$ and $U \subset V$ then $g_n(U) \subset g_n(V)$ for each $n \in N$.

S_3 For each $U \in \mathscr{T}$, $U = \bigcup_{i=1}^{\infty} g_n(U)$

g is called a stratification of X if g satisfies the above conditions.

[Theorem 3.1] A first countable cs-semistratifiable space is stratifiable.

Proof: Let g be a cs-semistratification for X suppose $p \in V$ where V is open. Let $\{W(n)\ n \in N\}$ be a base of neighborhood for p such that $V \supset W(1) \supset W(2) \supset \cdots$ If $W(n) \not\subset g_n(V)$ for each $n \in N$, choose points $y(n) \in W(n) - g_n(V)$ for each $n \subset N$. The sequence $\{y(n) | n \in N\}$ converges to p, and so there is a n_0 such that $\{y(n) | n \in N\}$ is eventually in $g_{n_0}(V)$. Therefore, for some $n \in N$, $W(n) \subset g_n(V)$, that is, $p \in g_n(V)$.

[corollary 3.2] A Frechét cs-semistratifiable space is c-semistratifiable.

Proof: Assume that K is a compact subset such that $\bigcap g_n(K) \neq K$. Then there exists a x such that $x \in \bigcap_n g_n(x) - K$. We can find a seqnence $\{x_n\}$ in K such that $x \in \bigcap_n g_n(x_n)$.

Let z be a cluster point of $\{x_n\}$ in K. Now the Frechétness of the space guarantees a subsequence $\{x_{ni}\}$ of $\{x_n\}$ which converges to z.

$\bigcap_{k \in N} g_k\ (\langle z:\ x_{n_i} \rangle) = \langle z : x_{n_i} \rangle \subset K$, which is a contradiction.

Note that the converse of thus corollary 3.2. does not hold. In fact, there exists a semistratifiable space which is not of Frechét.

[Lemma 3.3] A compact cs-semistratifiable space is metrizable.

Lemma 3.3 is shown by Sakong [7] and the Bing's metrization theorem. By Virture of the above terminology and the Martin's result [6], we gain the following theorem.

[Theorem 3.4] A compact cs-semistratifiable space X is complete.

Proof: Let X be a compact cs-semistratifable. Let $\langle a_1, a_2, \cdots\rangle$ be a cauchy sequence. We want to show that $\langle a_n\rangle$ converges a point x in X. $g_1(x)=\langle a_1, a_2, \cdots]$, $g_2(x)=\langle a_2, a_3, \cdots\}$, $g_3(x)=\{a_3, a_4, \cdots\}, \cdots i.e.$ $g_k(x)=\{a_n; n\geq k\}$. Thus $g_1(x)$ $g_2(x)\supset\cdots$ and the diameters of the $g_n(x)$ tend to zero since X is metrizable. Furthermore since $d(\overline{g_n(x)})=d(g_n(x))$ where $\overline{g_n(x)}$ is the closure $g_n(x)$, $\overline{g_1(x)}\supset\overline{g_2(x)}\supset\cdots$is a sequence of non-empty closed set whose diameters tend to zero. therefore, $\bigcap_n\overline{g_n(x)}\neq\phi$; $x\in\bigcap_n\overline{g_n(x)}$.

We claim that the cauchy sequence $\langle a_n\rangle$ converges to x. Let $\varepsilon>0$, since $\lim_{n\to\infty} d\ (\overline{g_n(x)})$ $=0$, $\exists\ n_0\in N$ such that $d\ (\overline{g_{n0}(x)})<\varepsilon$ and so $n>n_0\Rightarrow a_n,\ x\in\overline{g_{n_0}(x)}\Rightarrow d\ (a_n,\ x)\ <\varepsilon$. I_n other words, $\langle a_n\rangle$ converges to x. This completes the proof.

4. conclusion

As shown above we introduced the concept of cs-semistratifiable space and then we maked mention of some nice properties for cs-semistratifiable spaces.

Above all, It was happy that we related the classes of cs-semistratifiable spaces to the likely characterizations with the more familiar classes of spaces and that verified those theorem. To prove theorem 3.3, we could have recourse to the other theorem that every compact metric space is complete. But we would like to apply the concept of cs-semistratifiable space to show it.

By all account of cs-semistratifiable spaces, we must concede that theorems shown in this note are only a few fact of the iron in the fire for cs-semistratifiable spaces.

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