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COMMUTING INVOLVTIONS IN A LEFT ARTINIAN RING

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The involutions in a left Artinian ring A with identity are investigated.Those left Artinian rings A for which 2 is a unit in A and the set of involutions in A forms a finite abelian group are characterized by the number of involutions in A.

1. Introduction and basic definitions

Let A be a left Artinian ring with identity 1, let G denote the group of all units in A and let J denote the Jacobson radical of A.An involution in A is an element g in G such that $g^2 = 1$. An idempotent in A is an element e in A such that $e^2 = e$.Note that if 2 is a unit in A,then the mapping $e \to 1 - 2e$ is a bijection from the set of idempotents of A to the set of involutions of A.

Consequently, if 2 is a unit in A,then $ef = fe$ for all idempotents e and f in A if and only if $g_1 g_2 = g_2 g_1$ for all involutions g_1 and g_2 in A. Clealy, $g_1 g_2 = g_2 g_1$ for involutions g_1 and g_2 in A if and only if the set $\triangle$ of all involutions in A forms an abelian group under multiplication.

In [3], Cohen and Koh proved that if any ring with 1 has finite number of involutions in A,then the number is 1 or even,Wedderburn-Artinian proved that if A is a semisimple Artinian ring,then A is isomorphic to a direct product of a finite number of simple rings.Hence we obtain the following:

THEOREM 1.1. If A is a left Artinian ring with identity, then $A/J \cong \bigoplus_{i=1}^{n} M_i(D_i)$ where $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i for each $i = 1,2,\cdots,n$ and n is a positive integer.

2. Commuting involutions in a left Artinian ring

Recall that a ring A is said to be primitive if it has a faitful, irreducible A-module and an ideal P in a ring A is said to be primitive if A/P is a primitive ring. By the well-known fact that if a ring A has identity,then the A-module **M** is irreducible if and only if there exits a maximal left ideal I such that A/I and M are isomophic,in [2],the following theorem was proved:

THEORM 2.1. If a ring A has identity,then$(I\ ;\ A) = \{a \in A : aA \subseteq I\}$ is a primitive ideal if and only if I is a maximal.And if I is a left ideal of A, then$(I : A)$ is the largest ideal of A contained in A.

Proof. In[2], (pp52, Theorem 4)

In[2], the following theorem was also proved:

THEOREM 2.2. A primitive ideal of a ring A is prime.

Proof. In[2], (pp67, Theorem 6).

LEMMA 2.3. Let A be a left Artinian ring with identity such that $A/J \cong \bigoplus_{i=1}^{n} D_i$ where each D_i is a division ring of odd characteristic. If the set $\triangle$ of involutions in A forms an abelian group, then A has precisely 2^n involutions and n maximal left (or right)ideals.

Proof. Since $2+J$ is a unit in A/J, 2 is a unit in A.Therefore, the number of involutions in A is equal to the number of idempotents in A. By hypothesis,A/J has precisely 2^n idempotents,namely,$\{\bigoplus_{i=1}^{n} x_i \mid x_i = 0_i, 1_i\}$ where 0_i(*resp.* 1_i)is the additive identity (resp. multiplicative identity) of D_i.

Since each idempotent of A/J may be lifted to an idempotent of A,A has at least 2^n idempotents. Suppose that $|\triangle| > 2^n$.Then there exist distinct idempotents e and f in A such that $e + J = f + J$.Thus $e - f \in J$.Since J is a nilpotent ideal of A,there exits a positive integer m such that $(e - f)^m = 0$ ($m \geq 2$). Note that if m is odd,then $(e - f)^m = e - f$ and if m is even,then $e - 2ef + f = 0$,and so $e + f = 2ef$ – $(*)$.By multiplying e to both sides of $(*)$,we get $e = ef$. Thus $e + f = 2ef = 2e$ and so $e = f$,a contradiction.Hence A has precisely 2^n idempotents and hence 2^n involutions.

Next, for the simplicity of notion,assume that $A/J = \bigoplus_{i=1}^{n} D_i$ and let Π denote the canonical epimorphism of A onto A/J. For each $j = 1,2, \cdots,n$,let $A_j = \Pi^{-1}(\bigoplus_{i=1}^{n} B_i)$ where $B_j = \{0_j\}$ and $B_i = D_i$ for $i \neq j$. Note that $A_1, A_2 \cdots, A_n$ are maximal left (or right) ideals of A.Moreover, if I is any maximal left (or right) ideal of A,then $A_1 A_2 \cdots A_n \subseteq \bigcap_{i=1}^{n} A_i \subseteq J \subseteq I$.By theorem 2.1, $(I : A)$ is primitive and the largest ideal contained in A and so $A_1 A_2 \cdots A_n \subseteq (I : A)$.By theorem 2.2, $(I : A)$ is a prime ideal.Thus there exits an $i, 1 \leq i \leq n$,such that $A_i \subseteq I$.Consequently,$A_i = I$.So A has precisely n maximal left (or right) ideals.

THEOREM 2.4. Let A be a left Artinian ring with idenity such that $2\cdot 1$ is a unit in A,the set $\triangle$ of involutions in A is an abelian group under multiplication and $|\triangle|$ is finite.Let $m = |\triangle|$.Then $m = 2^n$ for some positive integer n and the following are equivalent:

(i) $m = 2^n$.

(ii) $A/J \cong \bigoplus_{i=1}^{n} D_i$ where each D_i is a division ring of odd characteristic.

(iii) A has precisely n maximal left (or right) ideals.

Proof. Since the order of each g in A dives 2, $|\triangle| = 2^n$ for some $n \geq 0$. Since $2\cdot 1$ is a unit in A,char(A) $\neq 2$ and so $n > 0$. By the preceding Lemma, (ii) implies (i) and (ii) implies (iii).

Assume (i) holds. Note that $A/J \cong \bigoplus_{i=1}^{n} M_i(D_i)$ where each $M_i(D_i)$ is the set of all the $n_i \times n_i$ matrices over a division ring D_i. Since the idempotents in A commute and for each idempotent in A/J, there exits an idempotents $\bar{e}$ in A such that $e + J = \bar{e}$ and if $\bar{e}_{ni}$ and $\bar{f}_{ni}$ are idempotents in $M_i(D_i)$, then $\bar{e}_{ni}\bar{f}_{ni} = \bar{f}_{ni}\bar{e}_{ni}$. So $n_i = 1$ for all $i=1,2,\cdots,k$, that is, $A/J \cong \bigoplus_{i=1}^{n} D_i$. Since $2\cdot 1$ is a unit in A,$2\cdot 1 + J$ is a unit in A/J. Therefore, each D_i is a division ring of odd characteristic

By Lemma 2.3, k = n. Hence (i) implies (ii).

Finally, assume (iii) holds. As above, $A/J \cong \bigoplus_{i=1}^{n} D_i$ where each D_i is a division ring of odd characteristic. Once again, Lemma 2.3 yields that $k = n$.

COROLLARY 2.5. Let A be a commutative Artinian ring with identity such that $2\cdot 1$ is a unit in A, the set $\triangle$ of involutions in A is an abelian group under multiplication and $|\triangle|$ is finite. Let $m = |\triangle|$. Then $m=2^n$ for some positive integer n and the following are equivalent:

(i) $m = 2^n$

(ii) $A/J \cong \bigoplus_{i=1}^{n} F_i$ where each F_i is a field of odd characteristic.

(iii) A has precisely n maximal ideals.

(iv) $A \cong \bigoplus_{i=1}^{n} A_i$ where each A_i is Artinian local ring with identity such that (A_i/J_i) is odd where for each i, J_i is the Jacobson radical of A_i.

Proof. By [4,Theorem IV.2.9],if A_i is an Artinian local ring with identity and char(A_i/J_i) is odd, then A_i has exactly 2 involutions. Therefore (iv) implies (i). Since A is a commutative Artinian ring

with idenitity and any Artinian ring is semiperfect, by [1, Proposition 7.6, and Theorem 27.6], $A \cong \oplus_{i=1}^{n} A_i$ where each A_i is an Artinian local ring with identity. Since $2 \cdot 1$ is a unit in A, char(A_i/J_i) is odd for all $i = 1,2 \cdots, m$. Thus A_i has exactly 2 involutions for all i [Theorem IV.2.9]. Since the number of involutions in $\oplus A_i$ is 2^n, $m = n$ if (i) holds.
So (i) implies (iv). Therefore (i) - (iv) are equivalent by Theorem 2.4.

As noted previously, the assumption that the set of all involutions in a ring A is equivalent to the property that $ef = fe$ for all idempotents e and f in A where 2 is a unit in A. We next classify left Artinian rings A with identity for which the idempotents of A commute.
For the simplicity of notation, assume $A/J = \oplus_{i=1}^{n} M_i(D_i)$ where each $M_i(D_i)$ is the set of all $n_i \times n_i$ matrices over a division ring D_i. Let Φ denote the canonical homomorphism of A onto A/J. For each i, let $M_j = \oplus_{i=1}^{n} B_i$ where $B_j = M_j$ and $B_j = \{0_i\}$ for $i \neq j$ and let $A_j = \Phi^{-1}(M_j)$.

COROLLARY 2.6. Let A be a left Artinian ring with identity such that 2 is a unit in A and A has precisely four involutions. Then A has precisely two maximal left ideals and $A/J \cong F_1 \times F_2$ where F_i is field of odd characteristic.

Proof. Since A has precisely four involutions and 2 is a unit in A, A has exactly four idempotents, 0, 1, e and $1 - e$ for some e in A. Since these idempotents commute, the involutions commute as well. Thus the corollary follows from Theorem 2.4.

THEOREM 2.7. Let A be a left Artinian ring with identity. The following are equivalent:

(i) For all idempotents e and f in A, $ef = fe$.

(ii) $A/J \cong \oplus_{i=1}^{n} D_i$ where each D_i is a division ring and n is a positive integer, A_i contains a unique nonzero idempotent ε_j such that if e is any idepotent in A, then $\varepsilon_j e = \varepsilon_j$ or $e\varepsilon_j = \varepsilon_j e = 0$.

Proof. (i)$\Rightarrow$(ii). Assume that (i) holds. Then if e and f are idempotents in A/J, then $ef = fe$. Consequently, $A/J \cong \oplus_{i=1}^{n} D_i$ where each D_i is a division ring. We may assume that $A/J = \oplus_{i=1}^{n} D_i$.

Let $e_j = (x_1, x_2, \cdots, x_n)$ where $x_j = 1_j$ and $x_i = 0_i$ for all $i \neq j$ and let ε_j be an idempotent in A such that $\varepsilon_j + J = e_j$. Then ε_j is a nonzero idempotent in A_j. For, let $f \in A_j$ be such that $f^2 = f$. Then $f + J = J$ or $f + J = \varepsilon_j + J$ Since the only idempotents in D_j are 1_j and 0_j. If $f + J = J$, then $f = 0$ since J contains no nonzero idempotent.

If $f + J = \varepsilon_j + J$, then $f - \varepsilon_j \in J$ since idempotents in A commute, $f - \varepsilon_j$ is idempotent of A and hence $f = \varepsilon_j$. Now let ε be an idempotent in A. Then $\varepsilon_j e$ is an idempotent in A, and so $\varepsilon_j e = 0 = e\varepsilon_j$ or $\varepsilon_j e = \varepsilon_j$.

(ii) ⇒ (i). Assume that (ii) holds. Let e and f be idempotents in A. We first show that $\varepsilon_i(ef - fe) = 0$ for each $i = 1,2,\cdots,n$. By assumption, $\varepsilon_i e = 0 = e\varepsilon_i$ or $\varepsilon_i e = \varepsilon_i$. If $\varepsilon_i e = 0$, then $\varepsilon_i(ef - fe) = \varepsilon_i fe = 0$, since $\varepsilon_i f = \varepsilon_i$ or $\varepsilon_i f = 0$. Suppose that $\varepsilon_i e = \varepsilon_i$. If $\varepsilon_i f = 0$, then as above, $\varepsilon_i(ef - fe) = 0$. Therefore, we may assume that $\varepsilon_i e = \varepsilon_i = \varepsilon_i f$. Then $\varepsilon_i(ef - fe) = \varepsilon_i f - \varepsilon_i e = \varepsilon_i - \varepsilon_i = 0$. Next note that for all $i, j, i \neq j, \varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i = 0$. For, if not, then $\varepsilon_i\varepsilon_j = \varepsilon_i$ and $\varepsilon_j\varepsilon_i = \varepsilon_i$ so $\varepsilon_i\varepsilon_j\varepsilon_i\varepsilon_j = \varepsilon_i\varepsilon_j$ which means that $\varepsilon_i\varepsilon_j$ is idempotent which is contained in J. Since J contains no nonzero idempotent, $\varepsilon_i\varepsilon_j = 0$, a contradiction. So $\{\varepsilon_i : i = 1,2,\cdots,n\}$ is a set of orthogonal idempotents in A. Note that for $j = 1,2,\cdots,n, \varepsilon_j + J = (0_1,\cdots,1_j,\cdots,0_n)$ and hence $\bigoplus_{i=1}^{n} \varepsilon_j + J = 1 + J$. So $1 - \bigoplus_{i=1}^{n} \varepsilon_j \in J$. Since $\{\varepsilon_i : i = 1,2,\cdots,n\}$ is a set of orthogonal idempo tents in A, $1 - \bigoplus_{i=1}^{n} \varepsilon_j$ is also idempotent. And again, $1 - \bigoplus_{i=1}^{n}\varepsilon_j = 0$, so $1 = \bigoplus_{i=1}^{n}\varepsilon_j$. Therefore, $(ef - fe) = 1(ef - fe) = (\bigoplus_{i=1}^{n} \varepsilon_j)$ $(ef - fe) = \bigoplus_{i=1}^{n} (\varepsilon_j(ef - fe)) = 0$. Thus, (ii) implies (i).

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