2-Normed Space에 관한 연구

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A Study on 2-Normed Spaces

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Abstract

The notion of a metric is to be regarded as a generalization of the notion of the distance between two points. The notion of 2-metric spaces is obtained by a generalization of the notion of area.

Unfortunately, the level of mathematics on 2 metric (or 2-normed) spaces is not so high, and the theory has not yet been developed until now. However, I think that this is a promising young branch in mathematics.

We mean a linear 2-normed space to be a pair $(L, \|\cdot, \cdot\|)$ where L is a linear space and $\|\cdot, \cdot\|$ is a real valued function defined on L such that for $x, y, z \in L$

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(2) $\|x, y\| = \|y, x\|$,

(3) For arbitrary real number α,

$$\|\alpha x, y\| = |\alpha| \|x, y\|,$$

(4) $\|x, y+z\| \leqq \|x, y\| + \|x, z\|$

$\|\cdot, \cdot\|$ is called a 2-norm.

Def 1. A sequence $\{x\}$ in a linear 2-normed space x is called a cauchy sequence, if there are y and z in x such that y and z are linearly independent.

$$\lim_{m,n}\|x_m - x_n, y\| = 0, \text{ and } \lim_{m,n}\|x_m - x_n, z\| = 0$$

Theorem 1. Let L be a linear 2-normed space.

a) If $\{x_n\}$ is a cauchy sequence in L with respect to x and y, then $\{\|x_n, x\|\}$ and $\{\|x, y\|\}$ are real cauchy sequences.

b) If $\{x_n\}$ and $\{y\}$ are cauchy sequence in L with respect to x and y, and $\{\beta_n\}$ is a real cauchy sequence, then $\{x_n + y_n\}$ and $\{\beta_n x_n\}$ are cauchy sequences in L.

Proof. a) $\|x_n, y\| = \|(x_n - x_m) + x_m, y\| \leqq \|x_n - x_m, y\| + \|x_m, y\|$

therefore $\|x_n, y\| - \|x_m, y\| \leqq \|x_n - x_m, y\|$.

Similarly, $\|x_m, y\| - \|x_n, y\| \leqq \|x_n - x_m, y\|$, that is $|\|x_n, y\| - \|x_m, y\|| \leqq \|x_n - x_m, y\|$.

Therefore $\{\|x_n, y\|\}$ is a real cauchy sequence since the lim $\|x_n - x_m, y\| = 0$.

Similarly, $\{||x_n, x||\}$ is a real cauchy sequence.

b) $||(x_n+y_n)-(x_m+y_m), x||=||(x_n-x_m)+(y_n-y_m), x|| \leqq ||x_n-x_m, x||+||y_n-y_m, x|| \to 0$

Similarly, $||(x_n+y_n)-(x_m+y_m), y|| \to 0$

Therefore $\{x_n+y_n\}$ is a cauchy sequence in L.

$$
\begin{aligned}
||\beta_n x_n-\beta_m x_m, x|| &= ||(\beta_n x_n-\beta_n x_m)+(\beta_n x_m-\beta_m x_m), x|| \\
&\leqq ||\beta_n x_n-\beta_n x_m, x||+||\beta_n x_m-\beta_m x_m, x|| \\
&= |\beta_n| \, ||x_n-x_m, x||+|\beta_n-\beta_m| \, ||x_m, x|| \\
&\leqq C_1 ||x_n-x_m, x||+C_2 |\beta_n-\beta_m| \to 0
\end{aligned}
$$

using the fact that $\{\beta_n\}$ and $\{||x_n, x||\}$ are real cauchy sequence and hence bounded.

Similarly, $||\beta_n x_n-\beta_m x_m, y|| \to 0$.

Therefore $\{\beta_n x_n\}$ is a cauchy sequence in L.

Def 2. A sequence $\{x_n\}$ in a linear 2-normed space x is called a convergent sequence,

If there is an x in X such that

$$\lim_m ||x_m-x, y||=0$$

for every y in X.

Def 3. A linear 2-normed space in which every Cauchy sequence in convergent is called a 2-Banach space.

Theorem 2. In any linear 2-normed space L:

a) If $x_n \to x$ and $y_n \to y$, then $x_n+y_n \to x+y$,

b) If $x_n \to x$ and $\beta_n \to \beta$, then $\beta_n x_n \to \beta x$,

c) If dim $L \geqq 2$, $x_n \to x$ and $x_n \to y$, then $x=y$.

Proof, a) $||(x_n+y_m)-(x+y), c||=||(x_n-x)+(y_n-y), c||$

$$\leqq ||x_n-x, c||+||y_n-y, c|| \to 0.$$

Therefore $x_n+y_n \to x+y$.

b) $||\beta_n x_n-\beta x, c||=||\beta_n x_n-\beta_n x+\beta_n x-\beta x, c||$

$$
\begin{aligned}
&\leqq ||\beta_n x_n-\beta_n x, c||+||\beta_n x-\beta x, c|| \\
&= |\beta_n| \, ||x_n-x, \beta||+|\beta_n-\beta| \, ||x, c|| \\
&= c||x_n-x, c||+|\beta_n-\beta| \, ||x, c||
\end{aligned}
$$

Using the fact that a real convergent sequence in bounded. Therefore $\beta_n x_n \to \beta x$ since the lim $||x_n-x, c||=0$ and the lim $|\beta_n-\beta|=0$.

c) $||x-y, c||=||(x_n-y)-(x_n-x), c||$

$$\leqq ||x_n-y, c||+||-(x_n-x), c||.$$

Therefore $||x-y, c||=0$ for all $c \in L$,

since $x_n \to x$ and $x_n \to y$. Hence $x-y$ and c are linearly dependent for all $c \in L$.

Since the dim $L \geqq 2$, the only way $x-y$ can be linearly dependent with all vectors $c \in L$, is for $x-y=0$.

Example 1. Let E_3 denote Euclidean vector three space. Let $x=x_1 i+x_2 j+x_3 k$ and $y=y_1 i+y_2 j+y_3 k$

$$\text{Define } ||x, y||=|x \times y|=\text{abs}\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$=|(x_2y_3-x_3y_2)i+(x_3y_1-x_1y_3)j+(x_1y_2-x_2y_1)k|$$

$$=[(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2+(x_1y_2-x_2y_1)^2]^{\frac{1}{2}}$$

Then $(E_3, ||\cdot,\cdot||)$ is a 2-Banach space.

Example 2. Let p_n denote the set of all real polynomials of degree $\leqq n$ on the interval $[0,1]$ We define addition and scalar multiplication in the usual way. Then p_n is a linear space over reals. Let $\{x_n\}$ $(n=1.2,\cdots\cdots,2n+1)$ be given $2n+1$ point in $[0.1]$.

For f, g, we put

$$||f,g||=\sum_i|f(x_i)\times g(x_i)|$$

If f, g are linearly independent, and $||f,g||=0$, if f, g are linearly dependent. Then p_n is a 2-Banach space.

On the other hand, there is a linear 2-normed space of dimension 3 which is not a 2-Banach space.

Example 3. Let E_3 denote Euclidean Vector three space where all coefficients are rationals, over the field of rationals. E_3 is a linear space.

Define $||\cdot,\cdot||$ in E_3 as in Example 1. Let

$x_n=\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}i$. $||x_n-x_m, i||=0$ hence the

$\lim||x_n-x_m, i||=0$. The

$$\lim||x_n-x_m, j||=\lim\left|\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}-\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}\right|=0$$

since $\left\{\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}\right\}$ is a real cauchy sequence.

Since i and j are linearly independent, $\{x_n\}$ is a cauchy sequence in E_3. Assume there is an $x=x_1j+x_2j+x_3k\epsilon E_3$ such that $x_n\to x$. Therefore the lime$||x_n-x, j||=0$,

that is, the

$$\lim\left[\left(\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}-x_1\right)^2+x_3^2\right]^{\frac{1}{2}}=0.$$

Clearly x_3 must be 0. Hence the $\lim\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}=x_1$. $\left\{\sum_{k=0}^{n}10^{\frac{-k(k+1)}{2}}\right\}$ converges in the real number system to an irrational number.

Therefore a must be irrational. Since E_3 is over the field of rationals, this is impossible.

Therefore E_3 is not a 2-Banach space.

But every 2-normed space of dimension 2 is a Banach space when the underlying field is complete.

Theorem 3. Every 2-normed space of dimension 2 is a 2-Banach space, when the underlying field is complete.

Proof, Let B be a linear 2-normed space with basis $\{e_1, e_2\}$. Let $\{x_n\}$ be a cauchy sequence in B. Therefore there exists linearly independent vector a and b in B such that the $\lim||x_n-x_m, a||=0$ and the $\lim||x_n-x_m, b||=0$.

Let $x_n = x_{n1}e_1 + x_{n2}e_2$, $a = a_1e_1 + a_2e_2$ and $b = b_1e_1 + b_2e_2$.

now $\|(x_n - x_m), a\| = \|(x_{n1} - x_{m1})e_1 + (x_{n2} - x_{m2})e_2, a_1e_1 + a_2e_2\| = |a_2(x_{n1} - x_{m1}) - a_2(x_{n2} - x_{m2})|\,\|e_1, e_2\|$.

Similarly $\|(x_n - x_m), b\| = |b_2(x_{n1} - x_{m1}) - b_1(x_{n2} - x_{m2})|\,\|e_1, e_2\|$.

Since e_1 and e_2 are linearly independent $\|e_1, e_2\| \neq 0$.

Therefore the $\lim|a_2(x_{n1} - x_{m1}) - a_1(x_{n2} - x_{m2})| = 0$ and the $\lim|b_2(x_{n1} - x_{m1}) - b_1(x_{n2} - x_{m2})| = 0$. Hence the

$\lim[a_2b_2(x_{n1} - x_{m1}) - a_1b_2(x_{n2} - x_{m2})] = 0$

and the

$$\lim[-a_2b_2(x_{n1} - x_{m1}) + a_2b_1(x_{n2} - x_{m2})] = 0.$$

Therefore by addition the $\lim(a_2b_1 - a_1b_2)(x_{n2} - x_{m2}) = 0$.

$a_2b_1 - a_1b_2 = 0$ implies $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ which is impossible, since a and b are linearly independent.

Hence, the $\lim|x_{n2} - x_{m2}| = 0$, that is, $\{x_{n2}\}$ is a cauchy sequence. Also, the $\lim[a_2b_1(x_{n1} - x_{m1}) - a_1b_1(x_{n2} - x_{m2})] = 0$ and the $\lim[-a_1b_2(x_{n1} - x_{m1}) + a_1b_1(x_{n2} - x_{m2})] = 0$.

Therefore by addition, the $\lim(a_2b_1 - a_1b_2)(x_{n1} - x_{m1}) = 0$.

Since $a_2b_1 - a_1b_2 \neq 0$, the $\lim|x_{n1} - x_{m1}| = 0$, that is, $\{x_{n1}\}$ is a cauchy sequence.

Since $\{x_{n1}\}$ and $\{x_{n2}\}$ are real cauchy sequences, there are real numbers y_1 and y_2 such that the $\lim x_{n1} = y_1$ and the $\lim x_{n2} = y_2$.

Let $x = y_1e_1 + y_2e_2$. Claim $x_n \to x$. Let $c = c_1e_1 + c_1e_2$ be an element of B. The $\lim\|(x_n - x), c\| = \lim\|(x_{n1} - y_1)e_1 + (x_{n2} - y_2)e_2, c_1e_1 + c_2e_2\| = \lim|c_2(x_{n1} - y_1) - c_1(x_{n2} - y_2)|\,\|e_1, e_2\| = 0$ since the $\lim x_{n1} = y_1$ and the $\lim x_{n2} = y_2$.

Therefore $x_n \to x$, that is, B is a 2-Banach space.

Next we shall explain a very important result about a 2-normed space.

Let X be a 2-normed space, and Let a be a given non-zero element of X. We denote the 1-dimensional linear space generated by a by $L(a)$. Then we can consider the quotient space $X/L(a)$. As well known, this space $X/L(a)$ is also a linear space:

For x in X, Let x a denote the equivalence class of x.

Then the addition and the scalar multiplication are given by

$$x_a + y_a = (x + y)_a, \quad \alpha x_a = (\alpha x)_a.$$

If $x_a = y_a$, then we have

$$|\,\|x, a\| - \|y, a\|\,| \leqq \|x - y, a\| = 0.$$

Hence, $\|x, a\| = \|y, a\| \cdots$. Therefore the real valued function $\|\cdot\|_a$ given by $\|x_a\|_a = \|x, a\|$ is welldefined. Then this new function is a norm on $X/L(a)$.

(1) $\|xa\|_a = 0$ if and only if $\|x, a\| = 0$, if and only if $x \in L(a)$, if and only if $x_a = 0$.

(2) $\|\alpha x_a\|_a = \|(\alpha x)_a\|_a = \|\alpha x, a\| = |\alpha|\,\|x, a\| = |\alpha|\,\|x_a\|_a$

(3) $\|x_a + y_a\|_a = \|(x + y)_a\|_a = \|x + y, a\| \leqq \|x, a\| + \|y, a\| = \|x_a\|_a + \|y_a\|_a$

Hence $X/L(a)$ is a normed space.

Theorem 4. Let X be a 2-normed space. For a non-zero element a in X, the quotient space $X/(a)$ is a normed space, where $L(a)$ is the 1-dimensional linear space generared by a.

Reference

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